

# CALCULATING WITH TOPOLOGICAL ANDRÉ-QUILLEN THEORY, I: HOMOTOPICAL PROPERTIES OF UNIVERSAL DERIVATIONS AND FREE COMMUTATIVE $S$ -ALGEBRAS

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**ABSTRACT.** We adopt the viewpoint that topological André-Quillen theory for commutative  $S$ -algebras should provide usable (co)homology theories for doing calculations in the sense traditional within Algebraic Topology. Our main emphasis is on homotopical properties of universal derivations, especially their behaviour in multiplicative homology theories. There are algebraic derivation properties, but also deeper properties arising from the homotopical structure of the free algebra construction  $\mathbb{P}_R$  and its relationship with extended powers of spectra. In the connective case in ordinary  $\text{mod } p$  homology, this leads to useful formulae involving Dyer-Lashof operations in the homology of commutative  $S$ -algebras. Although some of our results should be obtainable using stabilisation, our approach seems more direct. We also discuss a reduced free algebra construction  $\tilde{\mathbb{P}}_R$ .

## INTRODUCTION

Topological André-Quillen homology and cohomology theories for commutative  $S$ -algebras were introduced by Maria Basterra, building on ideas of Igor Kriz as well as algebraic André-Quillen theory. Subsequent work, both individually and jointly in various combinations, by Basterra, Gilmour, Goerss, Hopkins, Kuhn, Lazarev, Mandell, McCarthy, Minasian, Reinhard, Richter Robinson, Whitehouse as well as the present author, has laid out the basic structure and provided key relationships with other areas.

In this work we continue to adopt the viewpoint of [4], regarding TAQ as providing usable (co)homology theories for doing calculations in the sense traditional within Algebraic Topology.

Our main emphasis is on homotopical properties of universal derivations, especially their behaviour in multiplicative homology theories. As the name suggests, there are algebraic derivation properties, but also deeper properties arising out of the homotopical structure of the free algebra construction and its relationship with extended powers of spectra. In the connective case and in ordinary  $\text{mod } p$  homology, this leads to useful formulae involving Dyer-Lashof operations in the homology of commutative  $S$ -algebras. Some of our results are probably obtainable using stabilisation, but our approach seems more direct.

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*Date:* 11/09/2012 version 3

arXiv:1208.1868 .

*2010 Mathematics Subject Classification.* Primary 55P43; Secondary 13D03, 55N35, 55P48.

*Key words and phrases.*  $S$ -module,  $S$ -algebra, cell algebra, topological André-Quillen (co)homology, power operations.

Part of this work was carried out in the period 2007–8 when the author was supported by a YFF Norwegian Research Council grant while at the University of Oslo, and later while receiving an EPSRC research grant. The author would like to thank Maria Basterra, Marcel Bökstedt, Bob Bruner, Helen Gilmour, Tyler Lawson, Mike Mandell, Peter May, Birgit Richter, Constanze Roitzheim and John Rognes for helpful comments and encouragement over many years, and especially Philipp Reinhard for detailed comments and supplying the proof in Appendix A..

We also discuss a *reduced free algebra* construction  $\tilde{\mathbb{P}}_R$  which we learnt of from Tyler Lawson. This takes as input  $R$ -modules under a fixed cofibrant replacement for the  $R$ -module  $R$  and gives rise to a Quillen adjunction.

$$\begin{array}{ccc} & \tilde{\mathbb{P}}_R & \\ \mathcal{C}_R & \xrightleftharpoons{\quad} & S_R^0/\mathcal{M}_R \\ & \mathbb{U} & \end{array}$$

We will use this in a sequel to study spectral sequences related to those studied by Maria Basterra [7, section 5] and Haynes Miller [23].

We give some sample calculations, but our main concern is with laying the groundwork for future applications.

**Notation, etc.** When working over a fixed commutative ground ring  $\mathbb{k}$  such as  $\mathbb{F}_p$ , we often write  $\otimes$  for  $\otimes_{\mathbb{k}}$ ,  $\text{Hom}$  for  $\text{Hom}_{\mathbb{k}}$ , etc.

## 1. RECOLLECTIONS ON TAQ-THEORY

We will assume the reader is familiar with Basterra's foundational paper [7] and the further development of its ideas in [4]. All of this is founded on the notions of  $S$ -modules and commutative  $S$ -algebras of [12]. We briefly spell out some of the main ingredients.

If  $R$  is a commutative  $S$ -algebra, then its category of (left)  $R$ -modules  $\mathcal{M}_R$  is a model category and the category of commutative  $R$ -algebras  $\mathcal{C}_R$  consists of the commutative monoids in  $\mathcal{M}_R$  with monoidal morphisms. There is a free  $R$ -algebra functor  $\mathbb{P}_R: \mathcal{M}_R \rightarrow \mathcal{C}_R$  left adjoint to the forgetful functor  $\mathbb{U}: \mathcal{C}_R \rightarrow \mathcal{M}_R$ , and this pair gives a Quillen adjunction. We denote the derived (or homotopy) categories by  $\bar{\mathcal{H}}\mathcal{M}_R, \bar{\mathcal{H}}\mathcal{C}_R$ .

For a pair of commutative  $R$ -algebras  $A \rightarrow B$  there is a  $B$ -module  $\Omega_A(B)$  which is well defined up to isomorphism in  $\bar{\mathcal{H}}\mathcal{M}_B$ . This comes with a canonical morphism in  $\bar{\mathcal{H}}\mathcal{M}_A$ , the *universal derivation*

$$\delta_{(B,A)}: B \rightarrow \Omega_A(B),$$

characterised by a natural isomorphism

$$\bar{\mathcal{H}}\mathcal{C}_A/B(B, B \vee X) \cong \bar{\mathcal{H}}\mathcal{M}_B(\Omega_A(B), X),$$

where  $X \in \mathcal{M}_B$  and  $B \vee X$  denotes the *square zero extension* of  $B$  by  $X$  viewed as a  $B$ -algebra over  $B$ .

*Topological André-Quillen homology* and *cohomology* with coefficients in a  $B$ -module  $M$  are defined by

$$\begin{aligned} \text{TAQ}_*(B, A; M) &= \pi_*(M \wedge_B \Omega_A(B)), \\ \text{TAQ}^*(B, A; M) &= \pi_{-*}(F_B(\Omega_A(B), M)) = \bar{\mathcal{H}}\mathcal{M}_B(\Omega_A(B), X)^*, \end{aligned}$$

where

$$\bar{\mathcal{H}}\mathcal{M}_B(\Omega_A(B), X)^n = \bar{\mathcal{H}}\mathcal{M}_B(\Omega_A(B), \Sigma^n X).$$

When  $E$  is a (unital)  $B$  ring spectrum, the composition

$$(1.1) \quad \begin{array}{ccccccc} & & \theta & & & & \\ & \nearrow & & \searrow & & & \\ \pi_*(B) & \longrightarrow & E_*(B) & \xrightarrow{(\delta_{(B,A)})^*} & E_*(\Omega_A(B)) & \xrightarrow{\theta'} & E_*^B(\Omega_A(B)) = \text{TAQ}_*(B, A; E) \end{array}$$

is the *TAQ-Hurewicz homomorphism*.

In [4] we showed how this could be interpreted as a cellular theory for cellular commutative  $R$ -algebras. A key ingredient was the basic observation that for an  $R$ -module  $Z$ ,

$$\Omega_R(\mathbb{P}_R Z) \cong \mathbb{P}_R Z \wedge_R Z.$$

For completeness we give a proof of this in Appendix A.

In [4] we developed the theory of connective  $p$ -local commutative  $S$ -algebras along the lines of [5] for spectra, making crucial use of TAQ with coefficients in  $H\mathbb{F}_p$ . In both of those works, one important outcome was the ability to detect *minimal atomic* objects using the vanishing of the appropriate Hurewicz homomorphism in positive degrees.

## 2. HOMOTOPICAL PROPERTIES OF UNIVERSAL DERIVATIONS

Let  $A$  be a commutative  $S$ -algebra. As pointed out in [17], for a commutative  $A$ -algebra  $B$ , the universal derivation  $\delta_{(B,A)}: B \rightarrow \Omega_A(B)$  is a homotopy derivation in the sense of the following discussion.

Let  $R$  be a commutative  $S$ -algebra, let  $E$  be an  $R$  ring spectrum and let  $M$  be a left  $E$ -module in the sense that there is a morphism  $\mu: E \wedge_R M \rightarrow M$  in  $\bar{h}\mathcal{M}_R$  satisfying appropriate associativity and unital conditions. We denote the product on  $E$  by  $\varphi: E \wedge_R E \rightarrow E$ .

**Definition 2.1.** A morphism  $\partial: E \rightarrow M$  in  $\bar{h}\mathcal{M}_R$  is a *homotopy derivation* if the following diagram in  $\bar{h}\mathcal{M}_R$  commutes.

$$(2.1) \quad \begin{array}{ccccc} & E \wedge_R E & \xrightarrow{\varphi} & E & \\ & \swarrow I \wedge \partial \vee \partial \wedge I & & \searrow \partial & \\ E \wedge_R M \vee M \wedge_R E & & & & M \\ & \searrow I \vee \text{switch} & & \nearrow \mu & \\ & E \wedge_R M \vee E \wedge_R M & \xrightarrow{\text{fold}} & E \wedge_R M & \end{array}$$

Now let  $A$  be a commutative  $S$ -algebra and let  $B$  be a commutative  $A$ -algebra. Following the remarks at end of [17, section 3], we recall that the universal derivation  $\delta_{(B,A)}: B \rightarrow \Omega_A(B)$  is a morphism in the derived category of  $A$ -modules  $\bar{h}\mathcal{M}_A$  which is also a homotopy derivation in the sense that the following diagram commutes in  $\bar{h}\mathcal{M}_A$

$$(2.2) \quad \begin{array}{ccc} B \wedge_A B & \xrightarrow{\text{prod}} & B \\ \downarrow I \wedge \delta_{(B,A)} + \text{switch} \circ (\delta_{(B,A)} \wedge I) & & \downarrow \delta_{(B,A)} \\ B \wedge_A \Omega_A(B) & \xrightarrow{\text{mult}} & \Omega_A(B) \end{array}$$

where elements of  $\bar{h}\mathcal{M}_A(X, Y)$  are added in the usual way.

Now suppose that  $E$  is a commutative  $B$  ring spectrum; this implies that  $E$  is a  $B$ -module and there is a unit morphism of  $B$  ring spectra  $B \rightarrow E$  in  $\bar{h}\mathcal{M}_B$ . Then on smashing with

copies of  $E$ , (2.2) gives another commutative diagram

(2.3)

$$\begin{array}{ccccc}
E \wedge_A B \wedge_A E \wedge_A B & \xrightarrow{\text{switch}} & E \wedge_A E \wedge_A B \wedge_A B & \xrightarrow{\text{prod}} & E \wedge_A B \\
\downarrow I \wedge I \wedge I \wedge \delta_{(B,A)} + \text{switch} \circ (I \wedge \delta_{(B,A)} \wedge I \wedge I) & & \downarrow I \wedge I \wedge I \wedge \delta_{(B,A)} + I \wedge I \wedge \text{switch} \circ (\delta_{(B,A)} \wedge I) & & \downarrow \delta_{(B,A)} \\
E \wedge_A B \wedge_A E \wedge_A \Omega_A(B) & \xrightarrow{\text{switch}} & E \wedge_A E \wedge_A B \wedge_A \Omega_A(B) & \xrightarrow{\text{prod} \wedge \text{mult}} & E \wedge_A \Omega_A(B)
\end{array}$$

which shows that the commutative  $E_*$ -algebra  $E_*^A B = \pi_*(E \wedge_A B)$  admits the  $E_*$ -module homomorphism

$$(\delta_{(B,A)})_*: E_*^A B \longrightarrow E_*^A \Omega_A(B).$$

Of course  $E_*^A \Omega_A(B)$  is also a left  $E_*^A B$ -module since  $\Omega_A(B)$  is a left  $B$ -module. Composing  $(\delta_{(B,A)})_*$  with the natural homomorphism  $E_*^A \Omega_A(B) \longrightarrow E_*^B \Omega_A(B)$ , we obtain an  $E_*$ -module homomorphism

$$\Delta_{(B,A)}: E_*^A B \longrightarrow E_*^A \Omega_A(B) \longrightarrow E_*^B \Omega_A(B).$$

We also have an augmentation  $\varepsilon: E_*^A B \longrightarrow E_*$  induced by applying  $\pi_*(-)$  to the evident composition

$$E \wedge_A B \longrightarrow E \wedge_A E \longrightarrow E.$$

Clearly  $\varepsilon$  is a morphism of  $E_*$ -algebras.

**Lemma 2.2.**  $(\delta_{(B,A)})_*$  and  $\Delta_{(B,A)}$  are  $E_*$ -derivations, so for  $u, v \in E_*^A B$ ,

$$\begin{aligned}
(\delta_{(B,A)})_*(uv) &= u\delta_{(B,A)}(v) \pm v\delta_{(B,A)}(u), \\
\Delta_{(B,A)}(uv) &= \varepsilon(u)\Delta_{(B,A)}(v) \pm \varepsilon(v)\Delta_{(B,A)}(u),
\end{aligned}$$

where the signs are determined from the degrees of  $u, v$  with the usual sign convention. In particular, if  $u, v \in \ker \varepsilon: E_*^A B \longrightarrow E_*$ , then

$$\Delta_{(B,A)}(uv) = 0,$$

so  $\Delta_{(B,A)}$  annihilates non-trivial products.

*Proof.* This involves diagram chasing using the definitions. □

### 3. THE FREE COMMUTATIVE ALGEBRA FUNCTOR $\mathbb{P}_R$

For a  $R$ -module  $X$  there is a free commutative  $R$ -algebra

$$\mathbb{P}_R X = \bigvee_{j \geq 0} X^{(j)} / \Sigma_j.$$

When  $R = S$  or a localisation of  $S$ , we will set  $\mathbb{P} = \mathbb{P}_S$ .

If  $X$  is cofibrant as an  $R$ -module then  $\mathbb{P}_R X$  is cofibrant as an commutative  $R$ -algebra. The functor  $\mathbb{P}_R$  is left adjoint to the forgetful functor  $\mathbb{U}: \mathcal{C}_R \longrightarrow \mathcal{M}_R$ , so for  $A \in \mathcal{C}_R$ ,

$$\mathcal{C}_R(\mathbb{P}_R(-), A) \cong \mathcal{M}_R(-, A),$$

where  $A = \mathbb{U}A$  is regarded as an  $R$ -module. In fact,

$$(3.1) \quad \begin{array}{ccc} & \mathbb{P}_R & \\ \mathcal{C}_R & \xrightleftharpoons{\quad} & \mathcal{M}_R \\ & \mathbb{U} & \end{array}$$

is a Quillen adjunction [12].

As it is a left adjoint,  $\mathbb{P}_R$  preserves colimits, including pushouts. As cell and CW  $R$ -modules are defined as iterated pushouts, applying  $\mathbb{P}_R$  to the skeleta leads to cell or CW skeleta. To make this explicit, suppose that  $X$  is an  $R$ -module with CW skeleta  $X^{[n]}$ , and attaching maps

$$j_n: \bigvee_i S_R^n \longrightarrow X^{[n]},$$

where  $S_R^n = \mathbb{F}_R S^n$  is the cofibrant model for the sphere spectrum in  $\mathcal{M}_R$ . Setting  $D_R^n = \mathbb{F}_R D^n$ , the  $(n+1)$ -skeleton  $X^{[n+1]}$  is defined by the pushout diagram

$$\begin{array}{ccc} \bigvee_i S_R^n & \longrightarrow & X^{[n]} \\ \downarrow & \lrcorner & \downarrow \\ \bigvee_i D_R^n & \longrightarrow & X^{[n+1]} \end{array}$$

which induces the pushout diagram

$$\begin{array}{ccc} \mathbb{P}_R(\bigvee_i S_R^n) & \longrightarrow & \mathbb{P}_R X^{[n]} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{P}_R(\bigvee_i D_R^n) & \longrightarrow & \mathbb{P}_R X^{[n+1]} \end{array}$$

in  $\mathcal{C}_R$ . So we obtain a CW filtration on  $\mathbb{P}_R X$  with  $n$ -skeleton

$$(\mathbb{P}_R X)^{\langle n \rangle} = \mathbb{P}_R X^{[n]}.$$

By [4, proposition 1.6], in the homotopy category of  $\mathbb{P}_R X$ -modules  $\overline{h}\mathcal{M}_{\mathbb{P}_R X}$ ,

$$\Omega_R(\mathbb{P}_R X) \cong \mathbb{P}_R X \wedge_R X.$$

The universal derivation

$$\delta_{(\mathbb{P}_R X, R)} \in \overline{h}\mathcal{M}_R(\mathbb{P}_R X, \Omega_R(\mathbb{P}_R X)) = \overline{h}\mathcal{M}_R(\mathbb{P}_R X, \mathbb{P}_R X \wedge_R X)$$

has the homotopy derivation property expressed in the homotopy commutative diagram (2.1). Furthermore, it corresponds to the inclusion  $X \longrightarrow \mathbb{P}_R X \wedge_R X$  under the sequence of isomorphisms

$$(3.2) \quad \begin{aligned} \overline{h}\mathcal{C}_R/\mathbb{P}_R X(\mathbb{P}_R X, \mathbb{P}_R X \vee \Omega_R(\mathbb{P}_R X)) &\cong \overline{h}\mathcal{M}_{\mathbb{P}_R X}(\Omega_R(\mathbb{P}_R X), \Omega_R(\mathbb{P}_R X)) \\ &\cong \overline{h}\mathcal{M}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R X, \mathbb{P}_R X \wedge_R X) \\ &\cong \overline{h}\mathcal{M}_R(X, \mathbb{P}_R X \wedge_R X). \end{aligned}$$

We will describe  $\delta_{(\mathbb{P}_R X, R)}$  as a morphism in the homotopy category  $\overline{h}\mathcal{M}_R$  using this identification.

Suppose that  $X'$  is a second copy of  $X$ . A representative for the homotopy class of the pinch map  $p: X \rightarrow X \vee X'$  induces morphisms of commutative  $R$ -algebras

$$\begin{array}{ccccc}
& & \mathbb{P}_R X & & \\
& \swarrow \mathbb{P}_R p & & \searrow & \\
\mathbb{P}_R(X \vee X') & & & & \mathbb{P}_R X \vee \mathbb{P}_R X \wedge_R X' \\
& \searrow \cong & & \nearrow \cong & \\
& \mathbb{P}_R X \wedge_R \mathbb{P}_R X' & \longrightarrow & \mathbb{P}_R X \wedge_R (R \vee X') &
\end{array}$$

where  $R \vee X'$  and  $\mathbb{P}_R X \vee \mathbb{P}_R X \wedge_R X'$  are square zero extensions of  $R$  and  $\mathbb{P}_R X$  respectively, and the horizontal morphism kills the wedge summands  $(X')^{(r)}/\Sigma_r$  with  $r \geq 2$ . Restricting to the summand  $X$  in  $\mathbb{P}_R X$  we obtain the pinch map  $p$ , and applying the isomorphism of (3.2) we see that the resulting composition

$$\begin{array}{ccccc}
& & \delta & & \\
& \curvearrowright & & \curvearrowright & \\
\mathbb{P}_R X & \longrightarrow & \mathbb{P}_R X \vee \mathbb{P}_R X \wedge_R X' & \longrightarrow & \mathbb{P}_R X \wedge_R X'
\end{array}$$

agrees with the universal derivation  $\delta_{(\mathbb{P}_R X, R)}$ .

In the homotopy category of  $S$ -modules,  $\delta_{(\mathbb{P}_R X, R)}$  is equivalent to a coproduct of maps

$$\delta_{(\mathbb{P}_R X, R), n}: E\Sigma_n \ltimes_{\Sigma_n} X^{(n)} \longrightarrow (E\Sigma_{n-1} \ltimes_{\Sigma_{n-1}} X^{(n-1)}) \wedge X,$$

where we have of course identified  $X'$  with  $X$ . In fact these are the transfer maps  $\tau_{n-1,1}$  of [11, definition II.1.4], *i.e.*,

$$(3.3) \quad \delta_{(\mathbb{P}_R X, R), n} = \tau_{n-1,1}: E\Sigma_n \ltimes_{\Sigma_n} X^{(n)} \longrightarrow (E\Sigma_{n-1} \ltimes_{\Sigma_{n-1}} X^{(n-1)}) \wedge X.$$

The derivation property of  $\delta_{(\mathbb{P}_R X, S)}$  is just a consequence of the commutativity of the diagram (3.4) below. We will give a brief explanation of this.

For detailed accounts of the stable homotopy theory involved, see [11, 18, 21]. We remark that in [11, chapter II, p. 24], the pinch map used above is referred to as the ‘diagonal’ since in the stable category finite products and coproducts coincide. To ease notation and exposition, we take  $R = S$  and set  $\mathbb{P} = \mathbb{P}_S$ ; however the general case is similar.

Let  $E\Sigma_{m+n}$  be a free contractible  $\Sigma_{m+n}$ -space, and let  $Y$  be a  $\Sigma_{m+n}$ -spectrum for  $m, n \geq 1$ ; we are interested in the case where  $Y = X^{(m+n)}$ , the  $(m+n)$ -th smash power of  $X$ . The equivariant half smash product  $E\Sigma_{m+n} \ltimes Y$  is a free  $\Sigma_{m+n}$ -spectrum, and the evident inclusions of subgroups

$$\Sigma_m \times \Sigma_n \hookrightarrow \Sigma_{m+n} \longleftarrow \Sigma_{m+n-1}$$

induce morphisms of spectra

$$E\Sigma_{m+n} \ltimes_{\Sigma_m \times \Sigma_n} Y \longrightarrow E\Sigma_{m+n} \ltimes_{\Sigma_{m+n}} Y \longleftarrow E\Sigma_{m+n} \ltimes_{\Sigma_{m+n-1}} Y$$

on orbit spectra. There are also transfer maps

$$E\Sigma_{m+n} \ltimes_{\Sigma_{m+n}} Y \xrightarrow{\tau_{m,n}} E\Sigma_{m+n} \ltimes_{\Sigma_m \times \Sigma_n} Y$$

$$E\Sigma_{m+n} \ltimes_{\Sigma_{m+n}} Y \xrightarrow{\tau_{m+n-1,1}} E\Sigma_{m+n} \ltimes_{\Sigma_{m+n-1}} Y$$

associated with these inclusions of subgroups. We will use the double coset formula of [18, §IV.6]. We are in the situation of [18, theorem IV.6.3], and our first task is to identify representatives for the double cosets in

$$\Sigma_m \times \Sigma_n \backslash \Sigma_{m+n} / \Sigma_{m+n-1}.$$

An elementary exercise with cycle notation shows that the following are true:

- the elements of  $\Sigma_{m+n} / \Sigma_{m+n-1}$  are the distinct left cosets  $(r, m+n) \Sigma_{m+n-1}$  where  $1 \leq r \leq m+n-1$ , together with  $\Sigma_{m+n-1}$ ;
- by definition, the elements of  $\Sigma_m \times \Sigma_n \backslash \Sigma_{m+n} / \Sigma_{m+n-1}$  are the  $\Sigma_m \times \Sigma_n$ -orbits in  $\Sigma_{m+n} / \Sigma_{m+n-1}$  and these are represented by  $(m, m+n)$  and  $\text{id}$ . In fact the orbit of  $(m, m+n)$  contains all the  $(r, m+n)$  with  $1 \leq r \leq m$ , and the orbit of the identity  $I$  contains all of the transpositions  $(m+r, m+n)$  with  $1 \leq r \leq n$ .

It is straightforward to verify the identities

$$\Sigma_m \times \Sigma_{n-1} = \Sigma_m \times \Sigma_n \cap \Sigma_{m+n-1},$$

$$\Sigma_{m-1} \times \Sigma_n = \Sigma_m \times \Sigma_n \cap (m, m+n) \Sigma_{m+n-1} (m, m+n).$$

Now the double coset formula tells us that in the homotopy category, there is a commutative diagram having the following form.

(3.4)

$$\begin{array}{ccc}
 & E\Sigma_{m+n} \ltimes_{\Sigma_m \times \Sigma_n} Y & \\
 \swarrow \tau_{n-1,1} \vee \tau_{m-1,1} & & \searrow \\
 E\Sigma_{m+n} \ltimes_{\Sigma_m \times \Sigma_{n-1}} Y \vee E\Sigma_{m+n} \ltimes_{\Sigma_{m-1} \times \Sigma_n} Y & & E\Sigma_{m+n} \ltimes_{\Sigma_{m+n}} Y \\
 \downarrow & & \downarrow \tau_{m+n-1,1} \\
 E\Sigma_{m+n} \ltimes_{\Sigma_{m+n-1}} Y \vee E\Sigma_{m+n} \ltimes_{\Sigma_{m+n-1}} Y & \xrightarrow{\text{fold}} & E\Sigma_{m+n} \ltimes_{\Sigma_{m+n-1}} Y
 \end{array}$$

#### 4. POWER OPERATIONS AND THE FREE FUNCTOR $\mathbb{P}$

We will describe another result on the effect of certain transfer maps in homology that sheds light on the calculation of universal derivations. We begin by recalling some standard facts about the homology of extended powers.

Let  $p$  be a prime and let  $V = V_*$  be a graded  $\mathbb{F}_p$ -vector space. The inclusion  $C_p \leq \Sigma_p$  of the subgroup of cyclic permutations  $C_p = \langle \gamma \rangle$  with  $\gamma = (1, 2, \dots, p)$  has index  $(p-1)!$ , so the associated transfer homomorphism provides a splitting for the induced homomorphism in group homology with coefficients in the  $p$ -fold tensor power  $V^{\otimes p}$  with the obvious action.

$$\begin{array}{ccc}
 & \text{Tr}_{C_p}^{\Sigma_p} & \\
 & \curvearrowright & \\
 H_*(C_p; V^{\otimes p}) & \longrightarrow & H_*(\Sigma_p; V^{\otimes p})
 \end{array}$$

Furthermore, the homology of the subgroup  $\Sigma_{p-1} \leq \Sigma_p$  is trivial in positive degrees, *i.e.*,

$$H_*(\Sigma_{p-1}; V^{\otimes p}) = H_0(\Sigma_{p-1}; V^{\otimes p}) = (V^{\otimes p})_{\Sigma_{p-1}}.$$

Hence the associated transfer homomorphism is also zero in positive degrees, *i.e.*, for  $k > 0$ ,

$$(4.1) \quad 0 = \text{Tr}_{\Sigma_{p-1}}^{\Sigma_p} : H_k(\Sigma_p; V^{\otimes p}) \longrightarrow H_k(\Sigma_{p-1}; V^{\otimes p}).$$

In fact the diagram of subgroup inclusions

$$\begin{array}{ccc} 1 & \hookrightarrow & C_p \\ \downarrow & & \downarrow \\ \Sigma_{p-1} & \hookrightarrow & \Sigma_p \end{array}$$

induces a commutative diagram of split epimorphisms.

$$\begin{array}{ccc} V^{\otimes p} & \longrightarrow & H_*(C_p; V^{\otimes p}) \\ \text{Tr}_1^{\Sigma_{p-1}} \left( \begin{array}{c} \downarrow \\ (V^{\otimes p})_{\Sigma_{p-1}} \end{array} \right) & & \left( \begin{array}{c} \downarrow \\ H_*(\Sigma_p; V^{\otimes p}) \end{array} \right) \text{Tr}_{C_p}^{\Sigma_p} \\ (V^{\otimes p})_{\Sigma_{p-1}} & \longrightarrow & H_*(\Sigma_p; V^{\otimes p}) \end{array}$$

This can be generalised to  $\Sigma_{p^m}$  where  $m \geq 2$ . Then the  $p$ -order of  $|\Sigma_{p^m}|$  is

$$\text{ord}_p |\Sigma_{p^m}| = \frac{(p^m - 1)}{(p - 1)} = p^{m-1} + p^{m-2} + \cdots + p + 1.$$

Therefore the wreath product

$$\Sigma_p \wr \Sigma_{p^{m-1}} = \Sigma_p \ltimes (\overbrace{\Sigma_{p^{m-1}} \times \cdots \times \Sigma_{p^{m-1}}}^p) \leq \Sigma_{p^m}$$

has  $p$ -order

$$\begin{aligned} \text{ord}_p |\Sigma_p \wr \Sigma_{p^{m-1}}| &= 1 + p \frac{(p^{m-1} - 1)}{(p - 1)} \\ &= \frac{(p^m - 1)}{(p - 1)} \end{aligned}$$

so a transfer argument show that the inclusion induces a split epimorphism.

$$\begin{array}{ccc} & \xleftarrow{\text{Tr}_{\Sigma_p \wr \Sigma_{p^{m-1}}}^{\Sigma_{p^m}}} & \\ H_*(\Sigma_p \wr \Sigma_{p^{m-1}}; \mathbb{F}_p) & \longrightarrow & H_*(\Sigma_{p^m}; \mathbb{F}_p) \end{array}$$

Another calculation shows that

$$\text{ord}_p |\Sigma_{p^{m-1}}| = \frac{(p^m - 1)}{(p - 1)} - m = (p^{m-1} + p^{m-2} + \cdots + p + 1) - m$$

and

$$\begin{aligned} \text{ord}_p |\overbrace{\Sigma_{p^{m-1}} \times \cdots \times \Sigma_{p^{m-1}}}^{(p-1)} \times \Sigma_{p^{m-1}-1}| &= (p-1) \frac{(p^{m-1} - 1)}{(p - 1)} + \frac{(p^{m-1} - 1)}{(p - 1)} - (m - 1) \\ &= (p^{m-1} + p^{m-2} + \cdots + p + 1) - m \\ &= \text{ord}_p |\Sigma_{p^{m-1}}|. \end{aligned}$$

So

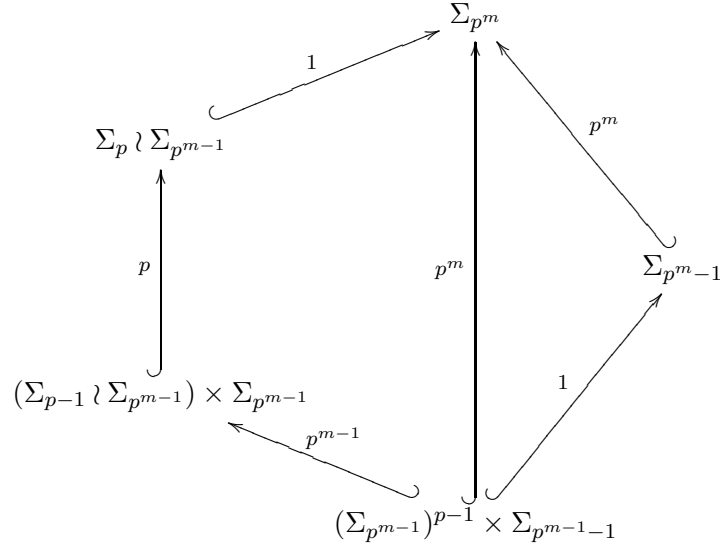
$$(\Sigma_{p^{m-1}})^{p-1} \times \Sigma_{p^{m-1}-1} \leq \Sigma_{p^{m-1}}$$

and these subgroups of  $\Sigma_{p^m}$  have the same  $p$ -order, hence the inclusion induces an isomorphism

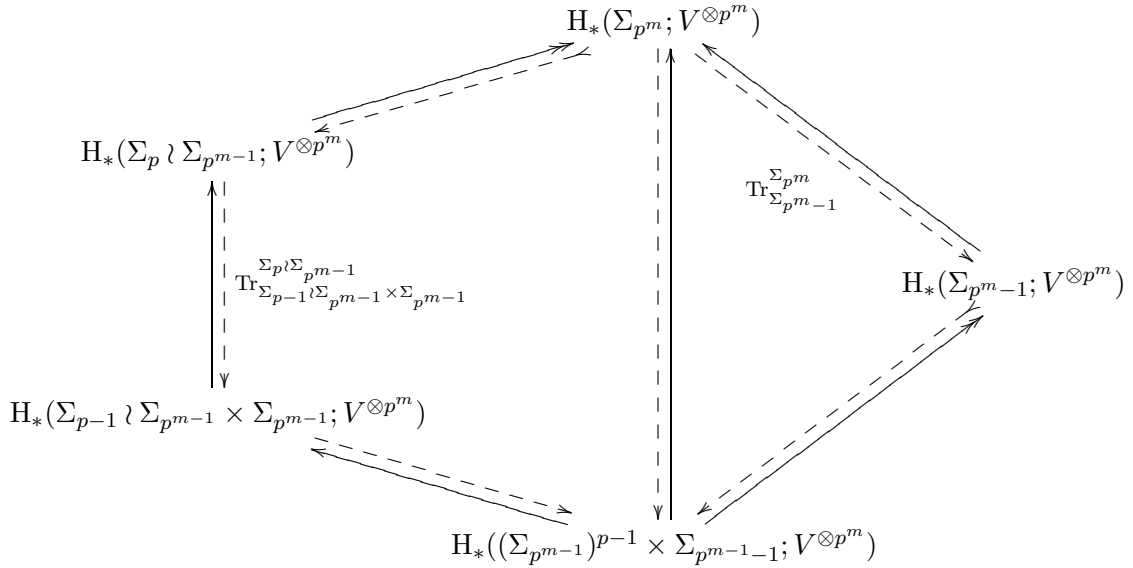
$$H_*((\Sigma_{p^{m-1}})^{p-1} \times \Sigma_{p^{m-1}-1}; \mathbb{F}_p) \xrightarrow{\cong} H_*(\Sigma_{p^{m-1}}; \mathbb{F}_p).$$



Consider the commuting diagram of subgroup inclusions



where the arrows are decorated with the  $p$ -power factors of the indices, *i.e.*, if  $H \leq G$  then the number would be  $p^{\text{ord}_p |G:H|}$ . Applying homology with coefficients in  $V^{\otimes p^m}$  with the evident action of  $\Sigma_{p^m}$ ,  $H_*(-; V^{\otimes p^m})$ , we obtain a commutative diagram of induced homomorphisms (solid arrows) and transfer homomorphisms (dashed arrows).



As the transfer is contravariantly functorial with respect to homomorphisms induced from inclusions, it is enough to show that  $\text{Tr}_{\Sigma_{p-1} \wr \Sigma_{p^{m-1}} \times \Sigma_{p^{m-1}}}^{\Sigma_{p^m}}$  is zero in positive degrees to deduce that the same holds for  $\text{Tr}_{\Sigma_{p^{m-1}}}^{\Sigma_{p^m}}$ . But this follows since

$$H_*(\Sigma_p \wr \Sigma_{p^{m-1}}; V^{\otimes p^m}) \cong H_*(\Sigma_p; H_*(\Sigma_{p^{m-1}}; V^{\otimes p^{m-1}})^{\otimes p})$$

and by (4.1) we already know the result for all transfer homomorphisms of the form

$$\text{Tr}_{\Sigma_{p-1}}^{\Sigma_p} : H_*(\Sigma_p; W^{\otimes p}) \longrightarrow H_*(\Sigma_{p-1}; W^{\otimes p})$$

for some  $\mathbb{F}_p$ -vector space  $W$ . To summarise, we have verified

**Lemma 4.1.** *For  $m \geq 1$ , the transfer*

$$\mathrm{Tr}_{\Sigma_{p^m-1}}^{\Sigma_{p^m}} : H_*(\Sigma_{p^m}; V^{\otimes p^m}) \longrightarrow H_*(\Sigma_{p^m-1}; V^{\otimes p^m})$$

*is zero in positive degrees.*

Recall the standard 2-periodic projective  $\mathbb{F}_p[C_p]$ -resolution of  $\mathbb{F}_p$ ,

$$(4.2) \quad \mathbb{F}_p \longleftarrow \mathbb{F}_p[C_p]e_0 \xleftarrow{1-\gamma} \mathbb{F}_p[C_p]e_1 \xleftarrow{1+\gamma+\dots+\gamma^{p-1}} \mathbb{F}_p[C_p]e_2 \xleftarrow{1-\gamma} \dots$$

where  $C_p = \langle \gamma \rangle$  with  $\gamma = (1, 2, \dots, p)$ . Tensoring over  $\mathbb{F}_p[C_p]$  gives a complex whose homology is  $H_*(C_p; V^{\otimes p})$ ,

$$0 \longleftarrow \mathbb{F}_p e_0 \otimes V^{\otimes p} \xleftarrow{1-\gamma} \mathbb{F}_p e_1 \otimes V^{\otimes p} \xleftarrow{1+\gamma+\dots+\gamma^{p-1}} \mathbb{F}_p e_2 \otimes V^{\otimes p} \xleftarrow{1-\gamma} \dots$$

where  $\otimes = \otimes_{\mathbb{F}_p}$ .

Let  $X$  be a connective cofibrant  $S$ -module, and let  $x \in H_n(X; \mathbb{F}_p)$  be a non-zero element. Recall the algebraic results of [19, lemma 1.4]. If  $p$  is a prime, then there are elements

$$e_r \otimes x^{\otimes p} = e_r \otimes \overbrace{x \otimes \dots \otimes x}^p$$

which survive to non-zero homology classes in  $H_*(C_p; H_*(X; \mathbb{F}_p)^{\otimes p})$  for  $r \geq 0$ , and where if  $p$  is odd,

- $n$  is even and  $r = 2s(p-1)$  or  $r = 2(s+1)(p-1) - 1$  for  $0 \leq s \in \mathbb{Z}$ ,
- $n$  is odd,  $r = (2s+1)(p-1)$  or  $r = (2s+1)(p-1) - 1$  for  $0 \leq s \in \mathbb{Z}$ .

These map to non-zero elements

$$\tilde{Q}_r x, \beta \tilde{Q}_r x \in H_*(\Sigma_p; H_*(X; \mathbb{F}_p)^{\otimes p})$$

depending on the parity of  $r$ . There is a canonical isomorphism

$$H_*(\Sigma_p; H_*(X; \mathbb{F}_p)^{\otimes p}) \xrightarrow{\cong} H_*(E\Sigma_p \ltimes_{\Sigma_p} X^{(p)})$$

and we also denote the images of  $\tilde{Q}_s x, \beta \tilde{Q}_s x$  by the same symbols. The natural weak equivalence

$$E\Sigma_p \ltimes_{\Sigma_p} X^{(p)} \xrightarrow{\sim} X^{(p)}/\Sigma_p$$

induces an isomorphism

$$\begin{array}{c} \xrightarrow{\cong} \\ H_*(\Sigma_p; H_*(X; \mathbb{F}_p)^{\otimes p}) \xrightarrow{\cong} H_*(E\Sigma_p \ltimes_{\Sigma_p} X^{(p)}; \mathbb{F}_p) \xrightarrow{\cong} H_*(X^{(p)}/\Sigma_p; \mathbb{F}_p) \end{array}$$

sending  $\tilde{Q}_r x$  to the element which we will denote by  $\overline{Q}_r x \in H_*(X^{(p)}/\Sigma_p; \mathbb{F}_p)$ . When  $p$  is odd, we will also write

$$\begin{array}{lll} \overline{Q}^t x = \overline{Q}_{(2t-n)(p-1)} x, & \beta \overline{Q}^t x = \overline{Q}_{(2t-n)(p-1)-1} x, & \text{if } n \text{ is even,} \\ \overline{Q}^t x = \overline{Q}_{(2t+1-n)(p-1)} x, & \beta \overline{Q}^t x = \overline{Q}_{(2t+1-n)(p-1)-1} x, & \text{if } n \text{ is odd,} \end{array}$$

in keeping with upper indexing for Dyer-Lashof operations; when  $p = 2$ , we set

$$\overline{Q}^t x = \overline{Q}_{t+n} x.$$

The action of the Dyer-Lashof operations  $Q^t, \beta Q^t$  on  $H_*(\mathbb{P}X; \mathbb{F}_p)$  described by Steinberger in [11, chapter III] is consistent with this notation; we will write  $Q^t \cdot x, \beta Q^t \cdot x$  when applying

such an operation to an element  $x \in H_*(X; \mathbb{F}_p) \subseteq H_*(\mathbb{P}X; \mathbb{F}_p)$  to avoid potential confusion when  $X$  is itself a commutative  $S$ -algebra.

**Lemma 4.2.** *For a connective cofibrant  $S$ -module  $X$ , and an element  $x \in H_*(X; \mathbb{F}_p)$ , under the natural map*

$$E\Sigma_p \ltimes_{\Sigma_p} X^{(p)} \xrightarrow{\sim} X^{(p)}/\Sigma_p \xrightarrow{\quad} \mathbb{P}X$$

$\rho$

in  $H_*(-; \mathbb{F}_p)$  we have

$$\rho_*(\overline{Q}^t x) = Q^t \cdot x, \quad \rho_*(\beta \overline{Q}^t x) = \beta Q^t \cdot x.$$

*Proof.* The basic observation is that a commutative  $S$ -algebra is an algebra over the monad  $\mathbb{P} \circ \mathbb{U}$  where the two model categories  $\mathcal{M}_S$  and  $\mathcal{C}_S$  are related by the Quillen adjunction of (3.1) with  $R = S$ .

$$\begin{array}{ccc} & \mathbb{P} & \\ \mathcal{C}_S & \xleftarrow{\quad} & \mathcal{M}_S \\ & \mathbb{U} & \end{array}$$

The definition of the Dyer-Lashof operations for a commutative  $S$ -algebra  $A$  involves the composition

$$E\Sigma_p \ltimes_{\Sigma_p} (A^c)^{(p)} \longrightarrow (A^c)^{(p)}/\Sigma_p \longrightarrow A^{(p)}/\Sigma_p \longrightarrow \mathbb{P}A \longrightarrow A$$

where  $(-)^c$  denotes cofibrant replacement in  $\mathcal{M}_S$ . When  $X$  is cofibrant in  $\mathcal{M}_S$  and  $A = \mathbb{P}X$ , we obtain a commutative diagram of the form

$$\begin{array}{ccccccc} E\Sigma_p \ltimes_{\Sigma_p} X^{(p)} & \xrightarrow{\quad} & & & & & \\ \downarrow & & \rho & & & & \\ E\Sigma_p \ltimes_{\Sigma_p} ((\mathbb{P}X)^c)^{(p)} & \longrightarrow & ((\mathbb{P}X)^c)^{(p)}/\Sigma_p & \longrightarrow & (\mathbb{P}X)^{(p)}/\Sigma_p & \longrightarrow & \mathbb{P}(\mathbb{P}X) \longrightarrow \mathbb{P}X \end{array}$$

and applying  $H_*(-; \mathbb{F}_p)$  to this gives the result.  $\square$

Using iterated extended powers associated with the subgroups

$$\underbrace{\Sigma_p \wr \cdots \wr \Sigma_p}_{\ell} \leq \Sigma_{p^\ell}$$

we can form elements

$$\overline{Q}^I x = \beta^{\varepsilon_1} \overline{Q}^{i_1} \cdots \beta^{\varepsilon_\ell} \overline{Q}^{i_\ell} x \in H_*(\mathbb{P}X; \mathbb{F}_p)$$

for  $x \in H_*(X; \mathbb{F}_p)$ , where  $I = (\varepsilon_1, i_1, \dots, \varepsilon_\ell, i_\ell)$ ,  $i_k > 0$  and  $\varepsilon_k = 0, 1$  (with  $\varepsilon_k = 0$  when  $p = 2$ ) can be interpreted in terms of the homology of iterated wreath powers, and we obtain the compatibility formula

$$\rho_*^\ell(\overline{Q}^I x) = Q^I \cdot x,$$

where  $\rho^\ell$  is defined in an obvious way from the following diagram.

$$\begin{array}{c}
\overbrace{E\Sigma_p \wr \cdots \wr \Sigma_p}^\ell \ltimes_{\Sigma_p \wr \cdots \wr \Sigma_p} X^{(p^\ell)} \\
\downarrow \\
E\Sigma_{p^\ell} \ltimes_{\Sigma_{p^\ell}} X^{(p^\ell)} \\
\downarrow \\
E\Sigma_{p^\ell} \ltimes_{\Sigma_{p^\ell}} ((\mathbb{P}X)^c)^{(p^\ell)} \longrightarrow ((\mathbb{P}X)^c)^{(p^\ell)} / \Sigma_{p^\ell} \longrightarrow (\mathbb{P}X)^{(p^\ell)} / \Sigma_{p^\ell} \longrightarrow \mathbb{P}(\mathbb{P}X) \longrightarrow \mathbb{P}X
\end{array}
\quad \xrightarrow{\rho^\ell}$$

We can now state an important result.

**Theorem 4.3.** *Let  $X$  be a connective cofibrant  $S$ -module. Then*

$$(\delta_{(\mathbb{P}X, S)})_* : H_*(\mathbb{P}X; \mathbb{F}_p) \longrightarrow H_*(\Omega_S(\mathbb{P}X); \mathbb{F}_p)$$

*annihilates every element of the form  $Q^I x$  where  $\text{length}(I) > 0$  and  $x \in H_*(X; \mathbb{F}_p)$ .*

*Proof.* This follows from the observation (3.3) together with Lemma 4.1.  $\square$

This generalises to give

**Theorem 4.4.** *Let  $A$  be a connective commutative  $S$ -algebra. Then*

$$(\delta_{(A, S)})_* : H_*(A; \mathbb{F}_p) \longrightarrow H_*(\Omega_S(A); \mathbb{F}_p)$$

*annihilates every element of the form  $Q^I a$  where  $\text{length}(I) > 0$  and  $a \in H_*(A; \mathbb{F}_p)$ . Hence the TAQ-Hurewicz homomorphism*

$$\theta' : H_*(A; \mathbb{F}_p) \longrightarrow H_*^A(\Omega_S(A); \mathbb{F}_p) = \text{TAQ}_*(A, S; H\mathbb{F}_p)$$

*also annihilates all such elements.*

*Proof.* Using the observation in the Proof of Lemma 4.2, we know there is a morphism of commutative  $S$ -algebras  $\mathbb{P}A \longrightarrow A$  extending the multiplication. Choose a cofibrant replacement  $A^c \longrightarrow A$  for the underlying  $S$ -module of  $A$ . By naturality there is a commutative diagram in the homotopy category of  $S$ -modules

$$\begin{array}{ccccc}
\mathbb{P}A^c & \longrightarrow & \mathbb{P}A & \longrightarrow & A \\
\delta_{(\mathbb{P}A^c, S)} \downarrow & & \delta_{(\mathbb{P}A, S)} \downarrow & & \delta_{(A, S)} \downarrow \\
\mathbb{P}A^c \wedge A^c & \longrightarrow & \Omega_S(\mathbb{P}A) & \longrightarrow & \Omega_S(A)
\end{array}$$

and on applying  $H_*(-) = H_*(-; \mathbb{F}_p)$  we obtain an algebraic commutative diagram.

$$\begin{array}{ccccc}
H_*(\mathbb{P}A^c) & \longrightarrow & H_*(\mathbb{P}A) & \longrightarrow & A \\
(\delta_{(\mathbb{P}A^c, S)})_* \downarrow & & (\delta_{(\mathbb{P}A, S)})_* \downarrow & & (\delta_{(A, S)})_* \downarrow \\
H_*(\mathbb{P}A^c \wedge A^c) & \longrightarrow & \Omega_S(\mathbb{P}A) & \longrightarrow & \Omega_S(A)
\end{array}$$

Since an element of the form  $Q^I a$  lifts back to an element  $Q^I a' \in H_*(\mathbb{P}A^c)$  as explained above, the result follows. The result about  $\theta'$  is immediate from the definition (1.1).  $\square$

## 5. THE REDUCED FREE COMMUTATIVE ALGEBRA FUNCTOR $\tilde{\mathbb{P}}_R$

Throughout, we fix a cofibrant commutative  $S$ -algebra  $R$ . The two model categories  $\mathcal{M}_R$  and  $\mathcal{C}_R$  are related by the Quillen adjunction

$$\begin{array}{ccc} & \mathbb{P}_R & \\ \mathcal{C}_R & \xleftarrow{\quad} & \mathcal{M}_R \\ & \mathbb{U} & \end{array}$$

where the right adjoint  $\mathbb{U}$  is the forgetful functor. For a cofibrant  $R$ -module  $Z$ , inclusion of the basepoint  $* \rightarrow X$  induces a cofibration of commutative  $R$ -algebras  $R = \mathbb{P}_R* \rightarrow \mathbb{P}_R Z$ , so  $\mathbb{P}_R Z$  is cofibrant in the model category  $\mathcal{C}_R$ . More generally a(n acyclic) cofibration  $f: X \rightarrow Y$  in  $\mathcal{M}_R$  induces a(n acyclic) cofibration  $\mathbb{P}_R f: \mathbb{P}_R X \rightarrow \mathbb{P}_R Y$  in  $\mathcal{C}_R$ .

In  $\mathcal{M}_R$ ,  $R$  is not cofibrant and we denote its functorial cofibrant replacement by  $S_R^0 = \mathbb{F}_R S^0$  and a weak equivalence induced by a map of spectra  $S_R^0 \rightarrow R$  which represents the unit.

$$\begin{array}{ccc} * & \xrightarrow{\quad} & S_R^0 \\ & \searrow & \downarrow \sim \\ & & R \end{array}$$

There is a unique induced morphism  $\mathbb{P}_R S_R^0 \rightarrow R$  in  $\mathcal{C}_R$ , but this need not be a cofibration. Using the functorial factorisation in  $\mathcal{C}_R$  we obtain a commutative diagram in  $\mathcal{C}_R$

$$\begin{array}{ccc} & \mathbb{P}_R S_R^0 & \\ \swarrow & & \searrow \\ \tilde{R} & \xrightarrow{\quad \sim \quad} & R \end{array}$$

which we use to fix the left hand arrow.

We will make use of the comma category  $S_R^0/\mathcal{M}_R$  of  $R$ -modules under  $S_R^0$ , whose objects are the morphisms  $S_R^0 \rightarrow X$  and whose morphisms are the commuting diagrams

$$\begin{array}{ccc} & S_R^0 & \\ \swarrow & & \searrow \\ X & \xrightarrow{\quad} & Y \end{array}$$

with initial object  $\text{Id}_{S_R^0}$  and terminal object  $S_R^0 \rightarrow *$ . This inherits a model structure from  $\mathcal{M}_R$ . Given  $i: S_R^0 \rightarrow X$  in  $S_R^0/\mathcal{M}_R$ , we obtain the induced morphism  $\mathbb{P}_R i: \mathbb{P}_R S_R^0 \rightarrow \mathbb{P}_R X$  in  $\mathcal{C}_R$ . If  $i$  is a cofibration, then  $i$  is cofibrant in  $S_R^0/\mathcal{M}_R$  and  $\mathbb{P}_R i: \mathbb{P}_R S_R^0 \rightarrow \mathbb{P}_R X$  is a cofibration in  $\mathcal{C}_R$ ; we will then write  $X/S_R^0$  for the cofibre of  $i$ . We obtain a pushout diagram of commutative  $R$ -algebras

$$\begin{array}{ccc} \mathbb{P}_R S_R^0 & \xrightarrow{\mathbb{P}_R i} & \mathbb{P}_R X \\ \downarrow & \lrcorner & \downarrow \\ \tilde{R} & \longrightarrow & \tilde{R} \wedge_{\mathbb{P}_R S_R^0} \mathbb{P}_R X \end{array}$$

and we set

$$\tilde{\mathbb{P}}_R X = \tilde{R} \wedge_{\mathbb{P}_R S_R^0} \mathbb{P}_R X.$$

If  $i^c: S_R^0 \rightarrow X^c$  is the cofibrant replacement of  $i$  in the comma category, then the pushout diagram of solid arrows in

$$\begin{array}{ccc}
\mathbb{P}_R S_R^0 & \xrightarrow{\mathbb{P}_R i^c} & \mathbb{P}_R X^c \\
\downarrow & \lrcorner & \downarrow \\
\tilde{R} & \xrightarrow{\quad} & \tilde{R} \wedge_{\mathbb{P}_R S_R^0} \mathbb{P}_R X^c
\end{array}
\quad
\begin{array}{ccc}
& \xrightarrow{\mathbb{P}_R i} & \mathbb{P}_R X \\
& \searrow & \downarrow \\
& & \tilde{R} \wedge_{\mathbb{P}_R S_R^0} \mathbb{P}_R X
\end{array}$$

defines the homotopy pushout of the first diagram. We will denote this by

$$\tilde{\mathbb{P}}_R^h X = \tilde{R} \wedge_{\mathbb{P}_R S_R^0} \mathbb{P}_R X^c.$$

Of course  $\tilde{\mathbb{P}}_R^h X$  is well-defined in the homotopy category  $\bar{h}\mathcal{C}_R$ , and we have in effect defined it by making functorial choices.

The model categories  $S_R^0/\mathcal{M}_R$  and  $\mathcal{C}_R$  are related by the Quillen adjunction

$$\begin{array}{ccc}
& \tilde{\mathbb{P}}_R & \\
\mathcal{C}_R & \xrightleftharpoons{\quad} & S_R^0/\mathcal{M}_R \\
& \tilde{\mathbb{U}} &
\end{array}$$

where the right adjoint  $\tilde{\mathbb{U}}$  is the forgetful functor sending  $A$  to the composition

$$S_R^0 \xrightarrow{\quad} R \xrightarrow{\quad} A$$

in  $\mathcal{M}_R$ . The total left derived functor of  $\tilde{\mathbb{P}}_R$  is  $\tilde{\mathbb{P}}_R^h$  and

$$\begin{array}{ccc}
& \tilde{\mathbb{P}}_R^h & \\
\bar{h}\mathcal{C}_R & \xrightleftharpoons{\quad} & \bar{h}(S_R^0/\mathcal{M}_R) \\
& \tilde{\mathbb{U}}^h &
\end{array}$$

is a derived adjunction on homotopy categories.

We already know that for cofibrant  $X$ , in the homotopy category  $\bar{h}\mathcal{M}_{\mathbb{P}_R X}$ ,

$$\Omega_R(\mathbb{P}_R X) \cong \mathbb{P}_R X \wedge_R X.$$

**Proposition 5.1.** *Let  $X \in S_R^0/\mathcal{M}_R$  be cofibrant. Then in the homotopy category  $\bar{h}\mathcal{M}_{\tilde{\mathbb{P}}_R X}$ ,*

$$\Omega_R(\tilde{\mathbb{P}}_R X) \cong \tilde{\mathbb{P}}_R X \wedge_R X/S_R^0.$$

*Proof.* First we recall some observations appearing in [4]. For a pushout diagram of cofibrations of commutative  $R$ -algebras

$$\begin{array}{ccc}
A & \xrightarrow{\quad} & B \\
\downarrow & \lrcorner & \downarrow \\
C & \xrightarrow{\quad} & B \wedge_A C
\end{array}$$

we have

$$\Omega_A(B \wedge_A C) \sim B \wedge_A \Omega_A(C)$$

by [7, proposition 4.6]. When  $A = \mathbb{P}_R Z$ , for a cofibrant  $R$ -module  $Z$ , and  $C = \mathbb{P}_R C Z$  (where  $CZ$  denotes the cone on  $Z$ ) with the natural inclusion  $\mathbb{P}_R Z \rightarrow \mathbb{P}_R C Z$ , this gives

$$(5.1) \quad \begin{aligned} \Omega_{\mathbb{P}_R Z}(B \wedge_{\mathbb{P}_R Z} \mathbb{P}_R C Z) &\sim B \wedge_{\mathbb{P}_R Z} \Omega_{\mathbb{P}_R Z}(\mathbb{P}_R C Z) \\ &\sim B \wedge_{\mathbb{P}_R Z} \mathbb{P}_R C Z \wedge_R \Sigma Z, \end{aligned}$$

where the last equivalence follows from [4, (1.16)].

Now take  $Z = S_R^0$ , assume that  $i: S^0 \rightarrow X$  is a cofibration, and also take  $B = \mathbb{P}_R X$  so that

$$\tilde{\mathbb{P}}_R X = \tilde{R} \wedge_{\mathbb{P}_R S_R^0} \mathbb{P}_R X.$$

We obtain a cofibre sequence of  $\tilde{\mathbb{P}}_R X$ -modules

$$\tilde{\mathbb{P}}_R X \wedge_{\mathbb{P}_R X} \Omega_R(\mathbb{P}_R X) \longrightarrow \Omega_R(\tilde{\mathbb{P}}_R X) \longrightarrow \Omega_{\mathbb{P}_R X}(\tilde{\mathbb{P}}_R X) \longrightarrow \tilde{\mathbb{P}}_R X \wedge_{\mathbb{P}_R X} \Sigma \Omega_R(\mathbb{P}_R X) \longrightarrow \dots$$

which is equivalent to

$$\tilde{\mathbb{P}}_R S_R^0 \wedge X \longrightarrow \Omega_R(\tilde{\mathbb{P}}_R X) \longrightarrow \tilde{\mathbb{P}}_R X \wedge \Sigma S_R^0 \longrightarrow \tilde{\mathbb{P}}_R S_R^0 \wedge \Sigma X \longrightarrow \dots$$

and so

$$\Omega_R(\tilde{\mathbb{P}}_R X) \cong \tilde{\mathbb{P}}_R X \wedge_R \Sigma X / S_R^0,$$

as required.  $\square$

Now let  $A$  be a commutative  $R$ -algebra. Composing the unit  $R \rightarrow A$  with the weak equivalence  $S_R^0 \rightarrow R$  we obtain the object

$$S_R^0 \xrightarrow{\sim} R \rightarrow A$$

in  $S_R^0/\mathcal{M}_R$ . Using functorial factorisation we obtain a cofibrant replacement

$$\begin{array}{ccc} & S_R^0 & \\ \swarrow & & \searrow \\ A^c & \xrightarrow{\sim} & A \end{array}$$

and so

$$\tilde{\mathbb{P}}_R^h A = \tilde{R} \wedge_{\mathbb{P}_R S_R^0} \mathbb{P}_R A^c.$$

**Remark 5.2.** The multiplication on a commutative  $R$ -algebra  $A$  extends to a morphism of commutative  $R$ -algebras  $\tilde{\mathbb{P}}_R A \rightarrow A$ . This follows from the evident commutative diagram of solid arrows

$$\begin{array}{ccc} \mathbb{P}_R S_R^0 & \longrightarrow & \mathbb{P}_R A \\ \downarrow \gamma & \lrcorner & \downarrow \\ \tilde{R} & \longrightarrow & \tilde{\mathbb{P}}_R A \\ \downarrow \sim & & \downarrow \text{dashed} \\ R & & A \end{array}$$

(Curved arrows from  $R$  to  $A$  represent the unit and extension of the product respectively.)

where the curved arrows come from the unit and extension of the product respectively.

## 6. THE ORDINARY HOMOLOGY OF FREE COMMUTATIVE $S$ -ALGEBRAS

For a commutative ring  $\mathbb{k}$ , and a graded  $\mathbb{k}$ -module  $V_*$ , we will write  $\mathbb{k}\langle V_* \rangle$  for the free commutative graded  $\mathbb{k}$ -algebra on  $V_*$ . If  $V_*$  is connective and  $V_0$  is a cyclic  $\mathbb{k}$ -module, we set  $\tilde{V}_* = V_*/V_0$ .

Let  $X$  be a cofibrant connective spectrum.

**Theorem 6.1.** *The rational homology of  $\mathbb{P}X$  is given by*

$$H_*(\mathbb{P}X; \mathbb{Q}) = \mathbb{Q}\langle H_*(X; \mathbb{Q}) \rangle.$$

In positive characteristic, the next result is fundamental. We use the standard convention for Dyer-Lashof monomials so that in an indexing sequence

$$I = (\varepsilon_1, i_1, \varepsilon_2, i_2, \dots, \varepsilon_\ell, i_\ell),$$

each  $i_r$  is positive, when  $p = 2$  all the  $\varepsilon_i$  are zero, while for odd  $p$ ,  $\varepsilon_i = 0, 1$ . The length of  $Q^I$  is  $\text{length}(Q^I) = \ell$ .

**Theorem 6.2.** *For  $p$  a prime,  $H_*(\mathbb{P}X; \mathbb{F}_p)$  is the free commutative graded  $\mathbb{F}_p$ -algebra generated by elements  $\overline{Q}^I x_j$ , where  $x_j$  for  $j \in J$  gives a basis for  $H_*(X; \mathbb{F}_p)$  and  $I = (\varepsilon_1, i_1, \varepsilon_2, \dots, \varepsilon_\ell, i_\ell)$  is admissible and satisfies  $\text{excess}(I) + \varepsilon_1 > |x_j|$ .*

So for  $p = 2$ , this gives the polynomial ring

$$H_*(\mathbb{P}X; \mathbb{F}_2) = \mathbb{F}_2[\overline{Q}^I x_j : j \in J, \text{excess}(I) + \varepsilon_1 > |x_j|].$$

Of course these results are very similar to those for the homology of  $\Omega^\infty \Sigma^\infty Z$  for a space  $Z$ ; for a convenient overview of the latter, see [20].

*Sketch of why Theorems 6.1 and 6.2 hold.* We learnt some of the following from Mike Mandell, see also [7, section 6].

Let  $R$  be a commutative  $S$ -algebra. The free functor  $\mathbb{P}_R$  for, sends  $R$ -modules to commutative  $R$ -algebras. As it is a left adjoint it preserves pushouts, so for  $R$ -modules  $X, Y$ ,

$$\mathbb{P}_R(X \vee Y) \cong \mathbb{P}_R(X) \wedge_R \mathbb{P}_R(Y).$$

For any commutative  $R$ -algebra  $A$ , base change gives

$$A \wedge_R \mathbb{P}_R(-) = \mathbb{P}_A(A \wedge_R (-)).$$

If  $R = S$  and  $H = H\mathbb{k}$  for a field  $\mathbb{k}$ , then

$$H \wedge \mathbb{P}(-) = \mathbb{P}_H(H \wedge (-)).$$

Applying  $\pi_*(-)$  gives a functor sending spectra to commutative graded  $\mathbb{k}$ -algebras,

$$X \mapsto H_*(\mathbb{P}(X)) = \pi_*(\mathbb{P}_H(H \wedge (-))),$$

which preserves pushouts, in particular it sends wedges to tensor products. For any spectrum  $X$ , as an  $H$ -module,  $H \wedge X$  is equivalent to a wedge of suspensions of  $H$ , so the calculation of  $H_*(\mathbb{P}(X)) = H_*(\mathbb{P}(X); \mathbb{k})$  reduces to that for spheres. For  $\mathbb{k} = \mathbb{Q}$  this gives the rational result.

When  $\mathbb{k} = \mathbb{F}_p$ , for a sphere  $S^n$  the answer is the free commutative  $\mathbb{k}$ -algebra on admissible Dyer-Lashof monomials  $Q^I$  applied to an element  $s_n$ , *i.e.*, elements of the form  $Q^I \cdot s_n = \overline{Q}^I s_n$ , where the indexing sequences  $I$  are admissible and satisfy  $\text{excess}(I) > |s_n| = n$ . Although this makes sense for  $n \in \mathbb{Z}$  (for the general case see [11, chapter III]), we only require the case where  $n \geq 0$ .



Using the ideas of Section 4, we know that for  $x \in H_*(X)$ , the element  $\overline{Q}^I x$  is the image under the canonical homomorphism

$$\begin{array}{ccc} H_*(E \ltimes_{\Sigma_{p^\ell}} X^{(p^\ell)}) & \xrightarrow{\cong} & H_*(X^{(p^\ell)}/\Sigma_{p^\ell}) \\ & \searrow & \downarrow \\ & & H_*(\mathbb{P}X) \end{array}$$

of an element obtained by forming iterated wreath powers of  $x$  in

$$H_*(E\Sigma_p \ltimes_{\Sigma_p} (E\Sigma_{p^k} \ltimes_{\Sigma_{p^k}} X^{(p^k)})).$$

Of course we can view  $\overline{Q}^I x$  as obtained from  $x$  by applying the Dyer-Lashof monomial

$$Q^I = \beta^{\varepsilon_1} Q^{i_1} \dots \beta^{\varepsilon_\ell} Q^{i_\ell}$$

which exists as an operation on the homology of any commutative  $S$ -algebra.  $\square$

We will describe the analogous results for  $\tilde{\mathbb{P}}X$ . Our main computational tool is the Künneth spectral sequence, and we will use its multiplicative properties and compatibility with the action of the Dyer-Lashof operations, see [9] for some related results on this.

We begin by stating the rational result whose proof we leave to the reader.

**Theorem 6.3.** *The rational homology of  $\tilde{\mathbb{P}}X$  is given by*

$$H_*(\tilde{\mathbb{P}}X; \mathbb{Q}) = \mathbb{Q}\langle \tilde{H}_*(X; \mathbb{Q}) \rangle.$$

The positive characteristic case is of course more interesting.

**Theorem 6.4.** *Let  $p$  be a prime. If  $X$  is a  $p$ -local Hurewicz spectrum, then  $H_*(\tilde{\mathbb{P}}X; \mathbb{F}_p)$  is the free commutative graded  $\mathbb{F}_p$ -algebra generated by elements  $\overline{Q}^I x_j$ , where  $x_j$  for  $j \in J$  gives a basis for  $\tilde{H}_*(X; \mathbb{F}_p) = H_*(X/S^0; \mathbb{F}_p)$  and  $I = (\varepsilon_1, i_1, \varepsilon_2, \dots, \varepsilon_\ell, i_\ell)$  is admissible and satisfies  $\text{excess}(I) + \varepsilon_1 > |x_j|$ .*

*Proof.* We set  $H_*(-) = H_*(-; \mathbb{F}_p)$ .

Since the pushout agrees with the smash product over  $\mathbb{P}S^0$ , there is a first quadrant Künneth spectral sequence with

$$E_{s,t}^2 = \text{Tor}_{s,t}^{H_*(\mathbb{P}S^0)}(H_*(\mathbb{P}X), \mathbb{F}_p) \implies H_{s+t}(\tilde{\mathbb{P}}X).$$

Here  $i'_*: H_*(\mathbb{P}S^0) \longrightarrow H_*(\mathbb{P}X)$  embeds the domain as a subalgebra since  $x_0 = i_*(s_0)$  generates  $H_0(X) = \mathbb{F}_p$  and

$$\begin{aligned} i'_*(s_0) &= x_0 - 1, \\ i'_*(\overline{Q}^I s_0) &= \overline{Q}^I(x_0) - \overline{Q}^I(1) = \overline{Q}^I(x_0) \quad \text{if } \text{length}(I) > 0. \end{aligned}$$

By the freeness of  $H_*(\mathbb{P}X)$ , it is a free  $i'_* H_*(\mathbb{P}S^0)$ -module, hence

$$\begin{aligned} E_{*,*}^2 &= \text{Tor}_{0,*}^{H_*(\mathbb{P}S^0)}(H_*(\mathbb{P}X), \mathbb{F}_p) \\ &= H_*(\mathbb{P}X) \otimes_{i'_* H_*(\mathbb{P}S^0)} \mathbb{F}_p \\ &= H_*(\mathbb{P}X) / (x_0 - 1, \overline{Q}^I x_0 : \text{length}(I) > 0). \end{aligned}$$

Thus the spectral sequence collapses at the  $E^2$ -term and the result follows.  $\square$

Armed with Theorems 4.3, we have

**Theorem 6.5.** *Let  $p$  be a prime and let  $X$  be a connective cofibrant spectrum. Then the universal derivation  $\delta_{(\mathbb{P}_X, S)}$  induces the derivation*

$$\Delta: H_*(\mathbb{P}X; \mathbb{F}_p) \longrightarrow \mathrm{TAQ}_*(\mathbb{P}X, S; H\mathbb{F}_p) = H_*(\mathbb{P}X; \mathbb{F}_p)$$

which acts on the  $\overline{Q}^l x$  with  $x \in H_*(X; \mathbb{F}_p)$  by the rule

$$\Delta(\overline{Q}^I x) = \begin{cases} x & \text{if } \text{length}(I) = 0, \\ 0 & \text{if } \text{length}(I) > 0. \end{cases}$$

More generally Theorem 4.4 gives

**Theorem 6.6.** *Let  $A$  be a cofibrant connective commutative  $S$ -algebra. Then the universal derivation  $\delta_{(A,S)}$  induces an  $\mathbb{F}_p$ -derivation*

$$\Delta: H_*(A; \mathbb{F}_p) \longrightarrow \mathrm{TAQ}_*(A, S; H\mathbb{F}_p)$$

which annihilates elements of the form  $Q^I a$  with  $a \in H_*(A; \mathbb{F}_p)$  for any Dyer-Lashof monomial  $Q^I$  with  $\text{length}(I) > 0$ .

**Example 6.7.** Consider the suspension spectrum  $\Sigma^{\infty-2}\mathbb{CP}^{\infty} = \Sigma^{-2}\Sigma^{\infty}\mathbb{CP}^{\infty}$ . A complex orientation for a ring spectrum  $E$  is the homotopy class of a map  $\Sigma^{\infty-2}\mathbb{CP}^{\infty} \rightarrow E$  such that the restriction

$$S = S^0 = \Sigma^{-2}\Sigma^\infty \mathbb{C}P^1 = \Sigma^{-2}\Sigma^\infty S^2 \longrightarrow E$$

is homotopic to the unit of  $E$ . when  $E$  is a commutative  $S$ -algebra, such a map  $\Sigma^{\infty-2}\mathbb{CP}^{\infty} \rightarrow E$  induces a unique morphism of commutative  $S$ -algebras

$$\mathbb{P}\Sigma^{\infty-2}\mathbb{C}P^{\infty} \longrightarrow E.$$

Because of the condition involving the bottom cell, there is a commutative diagram of solid arrows

$$\begin{array}{ccc}
\mathbb{P}S^0 & \longrightarrow & \mathbb{P}\Sigma^{\infty-2}\mathbb{C}P^\infty \\
\downarrow & \lrcorner & \downarrow \\
\mathbb{P}D^1 & \longrightarrow & \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty
\end{array}
\begin{array}{c}
\searrow \\
\cdots\searrow \\
\searrow
\end{array}
\begin{array}{c}
\\
\\
E
\end{array}$$

and hence a unique dotted arrow making the whole diagram commute. This shows that  $\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^{\infty}$  is universal for maps  $S^0 \rightarrow E$  which give complex orientations. Of course the inclusion map  $\Sigma^{\infty-2}\mathbb{C}P^{\infty} \rightarrow \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^{\infty}$  itself provides a complex orientation.

**Lemma 6.8.** *The universal complex orientation*

$$\Sigma^{\infty-2}\mathbb{C}P^{\infty} = \Sigma^{\infty-2}MU(1) \longrightarrow MU$$

induces a rational equivalence of commutative  $S$ -algebras  $\sigma: \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^{\infty} \longrightarrow MU$ . Furthermore, the inclusion map  $\Sigma^{\infty-2}\mathbb{C}P^{\infty} \longrightarrow \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^{\infty}$  induces a morphism of ring spectrum  $MU \longrightarrow \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^{\infty}$

$$MU \longrightarrow \widetilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^{\infty}$$

which provides a homotopy splitting of  $\sigma$  in  $\bar{h}_* \mathcal{M}_{MU}$ .

*Proof.* The rational result is straightforward since a Künneth spectral sequence argument gives

$$H_*(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty; \mathbb{Q}) = \mathbb{Q}[\tilde{\beta}_r : r \geq 1],$$

where  $\tilde{\beta}_r$  is the image of the canonical generator  $\beta_r \in H_{2r}(\mathbb{C}P^\infty)$ . Then the morphism  $\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty \rightarrow MU$  clearly induces an isomorphism of rings

$$H_*(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty; \mathbb{Q}) \rightarrow H_*(MU; \mathbb{Q}).$$

It is easy to see that

$$H_*(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty; \mathbb{Z}) \rightarrow H_*(MU; \mathbb{Z})$$

is epic.

The composition

$$\Sigma^{\infty-2}\mathbb{C}P^\infty \rightarrow MU \rightarrow \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty \xrightarrow{\sigma} MU$$

is homotopic to the canonical orientation, so the composition

$$MU \rightarrow \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty \xrightarrow{\sigma} MU$$

is homotopic to the identity by the classical universality of the commutative ring spectrum  $MU$  described by Adams [1].  $\square$

The morphism  $\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty \rightarrow MU$  can be converted into a fibration (in either of the two model categories  $\mathcal{M}_S$  or  $\mathcal{C}_S$ ), giving a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty & \xrightarrow{\sim} & (\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty)' \\ & \searrow & \swarrow \\ & MU & \end{array}$$

where  $(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty)'$  is cofibrant in  $\mathcal{C}_S$ . The map  $(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty)' \rightarrow MU$  is a morphism in the subcategory  $\mathcal{M}_{(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty)'}$  of  $\mathcal{M}_S$ .

**Corollary 6.9.** *The fibre of  $(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty)' \rightarrow MU$  is rationally trivial.*

A version of the next result appears in [6].

**Proposition 6.10.** *For a prime  $p$ , there can be no morphism of commutative  $S_{(p)}$ -algebras  $\theta: MU \rightarrow (\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty)_{(p)}$  for which  $\sigma \circ \theta$  is homotopic to the identity. Hence there can be no morphism of commutative  $S$ -algebras  $\theta: MU \rightarrow \tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty$  for which  $\sigma \circ \theta$  is homotopic to the identity.*

*Proof.* It suffices to prove the result for a prime  $p$ , and we will assume all spectra are localised at  $p$ . Assume such a morphism  $\theta$  existed. Then by naturality of  $\Omega_S$ , there are (derived) morphisms of  $MU$ -modules and a commutative diagram

$$\begin{array}{ccccc} & & \text{id} & & \\ & \nearrow & & \searrow & \\ \Omega_S(MU) & \xrightarrow{\theta_*} & \Omega_S(\tilde{\mathbb{P}}\Sigma^{\infty-2}\mathbb{C}P^\infty) & \xrightarrow{\sigma_*} & \Omega_S(MU) \end{array}$$

which induces a commutative diagram in  $\mathrm{TAQ}_*(-; H\mathbb{F}_p)$

$$\begin{array}{ccccc} & & \xrightarrow{\quad \mathrm{id} \quad} & & \\ H_*(\Sigma^2 ku; \mathbb{F}_p) & \xrightarrow{\theta_*} & H_*(\Sigma^{\infty-2} \mathbb{CP}_2^\infty; \mathbb{F}_p) & \xrightarrow{\sigma_*} & H_*(\Sigma^2 ku; \mathbb{F}_p) \end{array}$$

where  $\mathbb{CP}_2^\infty = \mathbb{CP}^\infty / \mathbb{CP}^1$ .

It is standard that

$$H_n(\Sigma^{\infty-2} \mathbb{CP}_2^\infty; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } n \geq 2 \text{ and is even,} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, when  $p = 2$ ,

$$H_*(ku; \mathbb{F}_2) = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \dots] \subseteq \mathcal{A}(2)_*$$

with  $|\zeta_s| = 2^s - 1$ , while when  $p$  is odd,  $\Sigma^2 ku \sim \bigvee_{1 \leq r \leq p-1} \Sigma^{2r} \ell$  with

$$H_*(\ell; \mathbb{F}_2) = \mathbb{F}_p[\zeta_1, \zeta_2, \zeta_3, \dots] \otimes \Lambda(\bar{\tau}_r : r \geq 2)$$

where  $|\zeta_s| = 2p^s - 2$  and  $|\bar{\tau}_s| = 2p^s - 1$ . Clearly this means that no such  $\theta$  can exist.  $\square$

At the prime 2,  $\Sigma^{\infty-2} \mathbb{CP}^\infty$  is known to be minimal atomic [5, proposition 5.9]. The next result shows that the functor  $\tilde{\mathbb{P}}$  need not preserve this property.  $^\ddagger$

**Proposition 6.11.** *The 2-local commutative  $S$ -algebra  $\tilde{\mathbb{P}}\Sigma^{\infty-2} \mathbb{CP}_{(2)}^\infty$  is not minimal atomic.*

*Proof.* If  $\tilde{\mathbb{P}}\Sigma^{\infty-2} \mathbb{CP}_{(2)}^\infty$  were minimal atomic then by [4, theorem 3.3], the TAQ Hurewicz homomorphism (induced from the universal derivation)

$$\theta : \pi_n(\tilde{\mathbb{P}}\Sigma^{\infty-2} \mathbb{CP}_{(2)}^\infty) \longrightarrow \mathrm{TAQ}_n(\tilde{\mathbb{P}}\Sigma^{\infty-2} \mathbb{CP}_{(2)}^\infty, S; H\mathbb{F}_2)$$

would be trivial for  $n > 0$ .

By naturality, there is a commutative diagram

$$\begin{array}{ccc} \pi_*(\tilde{\mathbb{P}}\Sigma^{\infty-2} \mathbb{CP}_{(2)}^\infty) & \xrightarrow{\sigma_*} & \pi_*(MU_{(2)}) \\ \theta \downarrow & & \downarrow \theta \\ \mathrm{TAQ}_n(\tilde{\mathbb{P}}\Sigma^{\infty-2} \mathbb{CP}_{(2)}^\infty, S; H\mathbb{F}_2) & \xrightarrow{\sigma_*} & \mathrm{TAQ}_n(MU_{(2)}, S; H\mathbb{F}_2) \\ \cong \downarrow & & \downarrow \cong \\ H_*(\mathbb{CP}_2^\infty; \mathbb{F}_2) & \xrightarrow{\sigma_*} & H_*(\Sigma^2 ku; \mathbb{F}_2) \end{array}$$

in which the surjectivity of the top row follows from Lemma 6.8. The 2-primary calculations of [4, section 5] show that the right hand Hurewicz homomorphism  $\theta$  is non-zero in positive degrees, hence so is the left hand one. Therefore  $\tilde{\mathbb{P}}\Sigma^{\infty-2} \mathbb{CP}_{(2)}^\infty$  cannot be minimal atomic.  $\square$

We leave the interested reader to formulate and verify analogues for an odd prime  $p$  based on desuspensions of the  $p$ -local summands of  $\Sigma^\infty \mathbb{CP}_{(p)}^\infty$ .

$^\ddagger$ For a while the author believed this was true and even claimed to have a proof during a talk!

**Remark 6.12.** We point out that  $\sigma_*: H_*(\mathbb{CP}_2^\infty; \mathbb{F}_2) \rightarrow H_*(\Sigma^2 ku; \mathbb{F}_2)$  is different from the homomorphism induced by any map of spectra  $\Sigma^{\infty-2} \mathbb{CP}_2^\infty \rightarrow \Sigma^2 ku$  which is an equivalence on the bottom cell. For such a map composed with the natural map  $\Sigma^2 ku \rightarrow H\mathbb{F}_2$  induces the homomorphism in homology given by

$$\Sigma^{-2} \beta_n \mapsto \begin{cases} \xi_s^4 & \text{if } n = 2^s, \\ 0 & \text{otherwise.} \end{cases}$$

But also  $\Sigma^{-2} \beta_n \mapsto b_{n-1}$  in  $H_*(MU; \mathbb{F}_2)$  and under the TAQ-Hurewicz homomorphism,

$$b_{n-1} \mapsto \begin{cases} \xi_s^2 & \text{if } n = 2^s + 1, \\ 0 & \text{otherwise.} \end{cases}$$

## 7. THE FREE COMMUTATIVE $S$ -ALGEBRA FUNCTOR AND $\Omega^\infty \Sigma^\infty Z$

Let  $Z$  be a connected based space. The infinite loop space  $\Omega^\infty \Sigma^\infty Z$  gives rise to a commutative  $S$ -algebra  $\Omega^\infty \Sigma_+^\infty Z$ , *i.e.*, the suspension spectrum of the based space  $\Omega^\infty \Sigma^\infty Z$  with a disjoint basepoint.

The natural (based) map  $Z \rightarrow \Omega^\infty \Sigma^\infty Z$  viewed as an unbased map induces a based map

$$\Sigma_+^\infty Z \rightarrow \Sigma_+^\infty \Omega^\infty \Sigma^\infty Z$$

which extends uniquely to a morphism of ring spectra

$$\mathbb{P}\Sigma_+^\infty Z \rightarrow \Sigma_+^\infty \Omega^\infty \Sigma^\infty Z.$$

The base points in  $Z$  and  $\Omega^\infty \Sigma^\infty Z$  pick out maps from the sphere  $S$  and there is a commutative diagram of commutative  $S$ -algebras

$$(7.1) \quad \begin{array}{ccc} \mathbb{P}S^0 & \xrightarrow{\quad} & \mathbb{P}\Sigma_+^\infty Z \\ \downarrow & \lrcorner & \downarrow \\ \tilde{S} & \dashrightarrow & \tilde{\mathbb{P}}\Sigma_+^\infty Z \\ \downarrow \sim & & \downarrow \text{dashed} \\ S & & \mathbb{P}\Sigma_+^\infty \Omega^\infty \Sigma^\infty Z \end{array}$$

(Note: The diagram shows curved arrows from  $\tilde{S}$  and  $S$  to  $\mathbb{P}\Sigma_+^\infty \Omega^\infty \Sigma^\infty Z$ , and a curved arrow from  $\mathbb{P}\Sigma_+^\infty Z$  to  $\mathbb{P}\Sigma_+^\infty \Omega^\infty \Sigma^\infty Z$ .)

where the acyclic fibration  $\tilde{S} \rightarrow S$  is that of Section 5.

**Proposition 7.1.** *For a connected based space  $Z$ , the morphism*

$$\tilde{\mathbb{P}}\Sigma^\infty Z \xrightarrow{\sim} \Sigma_+^\infty \Omega^\infty \Sigma^\infty Z$$

*of (7.1) is a weak equivalence.*

*Proof.* If  $\mathbb{k} = \mathbb{Q}$  and  $\mathbb{k} = \mathbb{F}_p$  for  $p$  a prime, comparison of the known answers for  $H_*(\tilde{\mathbb{P}}\Sigma^\infty Z; \mathbb{k})$  and  $H_*(\Omega^\infty \Sigma^\infty Z; \mathbb{k})$  shows that this morphism induces an isomorphism  $H_*(-; \mathbb{k})$ . It follows that it induces an isomorphism on  $H_*(-; \mathbb{Z})$ , hence it is a weak equivalence.  $\square$

## 8. SOME CALCULATIONS

Armed with our earlier results, we revisit some of the calculations of [4, section 5].

First we consider the TAQ-Hurewicz homomorphism for the  $E_\infty$  Thom spectrum  $MU$ . By work of Basterra and Mandell, in  $\bar{h}\mathcal{M}_{MU}$ ,

$$\Omega_S(MU) \cong MU \wedge \Sigma^2 ku.$$

At the prime  $p = 2$ , the Hurewicz homomorphism factors through  $H_*(MU; \mathbb{F}_2)$

$$\begin{array}{ccccccc} & & \theta & & & & \\ & \nearrow & & \searrow & & & \\ \pi_*(MU) & \longrightarrow & H_*(MU; \mathbb{F}_2) & \xrightarrow{\theta'} & \text{TAQ}_*(MU, S; \mathbb{H}\mathbb{F}_2) & \xrightarrow{\cong} & H_*(\Sigma^2 ku; \mathbb{F}_2) \end{array}$$

where

$$\begin{array}{ccc} H_*(MU; \mathbb{F}_2) & \xrightarrow{\theta'} & H_{*-2}(\Sigma^2 ku; \mathbb{F}_2) \\ \parallel & & \parallel \\ \mathbb{F}_2[b_r : r \geq 1] & & \Sigma^2 \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \dots] \end{array}$$

is a derivation and

$$\theta'(b_r) = \begin{cases} \Sigma^2 \xi_s^2 & \text{if } r = 2^s, \\ 0 & \text{otherwise.} \end{cases}$$

Here  $\zeta_s = \xi(\xi_s)$  is the conjugate of the Milnor generator in  $\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \dots] \subseteq \mathcal{A}(2)_*$ . This tells us that the elements  $b_{2^s}$  are not decomposable in terms of the Dyer-Lashof action, and recovers part of Kochman's result Theorem B.2.

For an odd prime  $p$ , the TAQ-Hurewicz homomorphism breaks up into  $(p-1)$  pieces corresponding to the Adams splitting of  $p$ -local connective  $K$ -theory, which gives

$$\Sigma^2 ku_{(p)} \sim \bigvee_{1 \leq r \leq p-1} \Sigma^{2r} \ell.$$

This gives an equivalent homomorphism

$$\theta' : H_*(MU; \mathbb{F}_p) \longrightarrow \bigoplus_{1 \leq r \leq p-1} H_*(\Sigma^{2r} \ell; \mathbb{F}_p).$$

Here

$$H_*(\ell; \mathbb{F}_p) = \mathbb{F}_p[\zeta_i : i \geq 1] \otimes \lambda_{\mathbb{F}_p}(\bar{\tau}_j : j \geq 2) \subseteq \mathcal{A}(p)_*,$$

and the component corresponding to  $\Sigma^{2r} \ell$  is determined in terms of generating functions by

$$\sum_{i \geq 0} b_i t^i \mapsto t^r \left( \sum_{j \geq 0} \xi_j t^{p^j - 1} \right)^r.$$

It follows that  $b_k \mapsto 0$  unless  $k \equiv r \pmod{p-1}$ . Write  $k = np^e$  with  $p \nmid n$ , so  $n \equiv r \pmod{p-1}$ . Now set

$$n - r = (p-1)(s_0 + s_1 p + \dots + s_d p^d)$$

with  $0 \leq s_i \leq p-1$  and  $s_d \neq 0$ . Then we obtain

$$b_{(s(p-1)+r)p^e} \mapsto \binom{r}{r-s_0, s_0-s_1, \dots, s_{d-1}-s_d, s_d} \xi_e^{r-s_0} \xi_{e+1}^{s_0-s_1} \dots \xi_{e+d-1}^{s_{d-2}-s_{d-1}} \xi_{e+d}^{s_{d-1}-s_d} \xi_{e+d+1}^{s_d}.$$

Notice that this can only give a non-zero answer if the following inequalities are satisfied:

$$1 \leq s_d \leq s_{d-2} \leq \dots \leq s_1 \leq s_0 \leq r.$$

In these cases  $b_{(s(p-1)+r)p^e}$  must be Dyer-Lashof indecomposable, and so we again recover Kochman's odd primary result of Theorem B.2.

Here is another example, the reader is invited to compare it with that of  $MU_{(2)}$  and  $MSp_{(2)}$  in [4, proposition 5.1].

**Proposition 8.1.** *The 2-local commutative S-algebra  $MSU_{(2)}$  is not minimal atomic.*

*Proof.* We recall that  $H_*(MSU; \mathbb{F}_2)$  is a polynomial algebra with a generator in each even degree greater than 2. There are many explicit generating families known, for example see [2, 3]. In fact,  $H_*(MSU; \mathbb{F}_2)$  can be identified as a subalgebra of  $H_*(MU; \mathbb{F}_2)$ , and then there are polynomial generators  $a_n \in H_{2n}(MU; \mathbb{F}_2)$  so that

$$H_*(MSU; \mathbb{F}_2) = \mathbb{F}_2[a_{2^s}^2 : s \geq 0] \otimes \mathbb{F}_2[a_{2^s k} : s \geq 0, k > 1 \text{ odd}] \subseteq H_*(MU; \mathbb{F}_2).$$

We will write  $a'_n$  the generator in degree  $2n$  where  $n \geq 2$ , for our purposes it is not important which choice we make here.

By [15, theorem 19(a)], the Dyer-Lashof indecomposables in  $H_*(MSU; \mathbb{F}_2)$  are the algebra generators appearing in degrees of the form  $2^m + 2^n$  where  $m, n \geq 0$ ; this includes the case  $2^s = 2^{s-1} + 2^{s-1}$  where  $s \geq 1$ .

Since there is a weak equivalence of infinite loop spaces  $BSU \sim \Omega^\infty \Sigma^4 ku$ , by [8],

$$\Omega_S(MSU) \cong MSU \wedge \Sigma^4 ku.$$

Therefore the TAQ-Hurewicz homomorphism factors as

$$\begin{array}{ccccccc} & & \theta & & & & \\ & \nearrow & & \searrow & & & \\ \pi_*(MSU) & \longrightarrow & H_*(MSU; \mathbb{F}_2) & \xrightarrow{\theta'} & \text{TAQ}_*(MSU, S; H\mathbb{F}_2) & \xrightarrow[\cong]{} & H_*(\Sigma^4 ku; \mathbb{F}_2) \end{array}$$

and in fact using the geometrically defined generators described in [2] it can be shown that

$$\text{im } \theta' = \mathbb{F}_2\{\Sigma^4 \xi_m^2 \xi_n^2 : m, n \geq 1\}.$$

Here  $\theta'$  has the effect

$$a'_{2n+1} \mapsto \Sigma^4 \xi_n^4 \quad (n \geq 0), \quad a'_{2m+2n} \mapsto \Sigma^4 \xi_m^2 \xi_n^2 \quad (n > m \geq 0).$$

However, this alone does not give us the result. We would like to use [4, theorems 3.2, 3.4], so we must show that

$$\theta: \pi_n(MSU) \longrightarrow \text{TAQ}_n(MSU, S; H\mathbb{F}_2)$$

is non-trivial for some  $n > 0$ . For this we will use work of Pengelley [24] on the Adams spectral sequence for  $\pi_n(MSU_{(2)})$ . In [24, theorem 2.6] it is shown that there are polynomial generators  $y'_{8k} \in H_{8k}(MSU; \mathbb{F}_2)$  for which the Adams differential  $d_2$  satisfies

$$d_2 y'_{8k} = \begin{cases} hq'_{s-1} \neq 0 & \text{if } k = 2^s, \\ 0 & \text{if } k \text{ is not a power of 2,} \end{cases}$$

and furthermore all trivial higher differentials in the spectral sequence are trivial. For our purposes what matters here is that each generator  $y'_{2m+3+2n+3}$  where  $n > m \geq 0$  is in the image of the classical Hurewicz homomorphism and under the TAQ-homomorphism it maps to  $\Sigma^4 \xi_{m+2} \xi_{n+2} \neq 0$ . This means that  $MSU_{(2)}$  cannot be minimal atomic.  $\square$

If the summand  $BoP$  of  $MSU_{(2)}$  were to have a commutative  $S$ -algebra structure, then Pen-  
gelley's results would imply that the mod2 TAQ-Hurewicz homomorphism was trivial, hence  
 $BoP$  would be minimal atomic. However, this depends on the observation that the classical  
mod2 Hurewicz homomorphism is trivial so we already know it is minimal atomic as a spec-  
trum [5] and hence it would be as a commutative  $S$ -algebra. So the use of TAQ would not be  
really necessary.

Here are some more examples.

**Example 8.2.** Let  $p$  be a prime and set  $H = H\mathbb{F}_p$ ,  $H_*(-) = H_*(-; \mathbb{F}_p)$ . Then TAQ-Hurewicz  
homomorphism

$$\theta': H_*(H) \longrightarrow \text{TAQ}_*(H, S; H)$$

has the following effect on

$$H_*(H) = \mathcal{A}(p)_* = \begin{cases} \mathbb{F}_p[\xi_i : i \geq 1] \otimes \Lambda(\tau_j : j \geq 0) & \text{if } p \text{ is odd,} \\ \mathbb{F}_2[\xi_i : i \geq 1] & \text{if } p = 2. \end{cases}$$

When  $p$  is odd,

$$\theta'(\tau_0) \neq 0, \quad \theta'(\tau_i) = \theta'(\xi_i) = 0 \quad (i \geq 1).$$

When  $p = 2$ ,

$$\theta'(\xi_1) \neq 0, \quad \theta'(\xi_i) = 0 \quad (i \geq 2).$$

The vanishing results follows from Steinberger's calculations of Dyer-Lashof operations in [11,  
chapter III, theorem 2.3]. The non-triviality results use the fact that the unit  $S \rightarrow H$  is 0-  
connected, hence by Basterra [7, lemma 8.2],  $\Omega_S(H)$  is 0-connected, see also [4, corollary 1.3].

Next we will consider the case of  $MO$ . The infinite loop space  $BO$  has Thom spectrum  $MO$   
which admits the structure of an  $E_\infty$  ring spectrum or equivalently of a commutative  $S$ -algebra.  
By Thom's theorem, this is known to split as a wedge of suspensions of  $H = H\mathbb{F}_2$  even as a  
ring spectrum

$$MO \sim \bigvee_{\alpha} \Sigma^{\alpha} H.$$

But as we will see, no such splitting can happen in  $\overline{h}\mathcal{C}_S$  because of obstructions lying in TAQ.  
Here the underlying infinite loop space is  $BO$  and the associated spectrum is  $ko\langle 1 \rangle$ , the 0-  
connected cover of  $ko$ . In the above splitting, the generalized Eilenberg-Mac Lane ring spectrum  
on the right hand side realises the graded polynomial ring

$$(8.1) \quad MO_* = \pi_*(MO) = \mathbb{F}_2[z_n : n \geq 1 \text{ is not of the form } 2^s - 1],$$

where  $z_n$  has degree  $n$ . For more on such ring spectra, see [10]. Let  $\underline{h}: \pi_n(MO) \rightarrow H_n(MO)$  de-  
note the usual mod 2 homology Hurewicz homomorphism. By Thom's theorem,  $\underline{h}$  is a monomor-  
phism and for the polynomial generators  $z_n$  of (8.1), the Hurewicz images  $\underline{h}(z_n)$  form part of a  
set of polynomial generators for  $H_*(MO)$  which has one generator in each positive degree.

By a result of Basterra and Mandell [8],

$$\Omega_S(MO) = MO \wedge ko\langle 1 \rangle,$$

where  $ko\langle 1 \rangle$  is the 0-connected cover of  $ko$ , defined by the cofibre sequence of  $ko$ -modules

$$ko\langle 1 \rangle \longrightarrow ko \longrightarrow H\mathbb{Z} \longrightarrow \Sigma ko\langle 1 \rangle.$$



On applying mod2 homology  $H_*(-)$  we obtain a short exact sequence

$$0 \rightarrow H_*(\Sigma^{-1}ko) \longrightarrow H_*(\Sigma^{-1}H\mathbb{Z}) \longrightarrow H_*(ko\langle 1 \rangle) \rightarrow 0$$

from which we deduce that as an  $H_*ko$ -module,

$$(8.2) \quad H_*(ko\langle 1 \rangle) = H_*(ko)\{\Sigma^{-1}\zeta_1^2, \Sigma^{-1}\zeta_2, \Sigma^{-1}\zeta_1^2\zeta_2\},$$

i.e., the free  $H_*(ko)$ -module on the generators  $\Sigma^{-1}\zeta_1^2, \Sigma^{-1}\zeta_2, \Sigma^{-1}\zeta_1^2\zeta_2$  in degrees 1, 2, 4 respectively.

We will make use of the TAQ-Hurewicz homomorphism

$$\theta: \pi_n(MO) \longrightarrow \text{TAQ}_n(MO, S; \mathbb{F}_2) = H_n(ko\langle 1 \rangle),$$

and so we need to understand the mod 2 homology  $H_*(ko\langle 1 \rangle)$ . In the dual Steenrod algebra

$$\mathcal{A}(2)_* = H_*(H) = \mathbb{F}_2[\xi_r : r \geq 1] = \mathbb{F}_2[\zeta_r : r \geq 1],$$

each generator  $\xi_r \in \mathcal{A}(2)_{2^r-1}$  is in the image of the natural map

$$H_{2^r}(\mathbb{RP}^\infty) \longrightarrow H_{2^r}(\Sigma H) = \mathcal{A}(2)_{2^r-1},$$

and  $\zeta_r = \chi(\xi_r)$ , the Hopf-algebra conjugate of  $\xi_r$ .

Now since  $\pi_1(ko\langle 1 \rangle) = \mathbb{F}_2$ , there is a canonical non-trivial homotopy class  $\psi: ko\langle 1 \rangle \longrightarrow \Sigma H$  inducing an isomorphism on  $\pi_1(-)$ . The horizontal composition in the diagram

$$\begin{array}{ccccc} H\mathbb{Z} & \longrightarrow & \Sigma ko\langle 1 \rangle & \xrightarrow{\psi} & \Sigma^2 H \\ & \searrow \text{reduction mod 2} & & \nearrow Sq^2 & \\ & & H & & \end{array}$$

factors as shown. In order to calculate the effect of the  $H_*(ko)$ -module homomorphism

$$\psi_*: H_*(ko\langle 1 \rangle) \longrightarrow H_{*-1}(H),$$

we first note that for  $r = 1, 2$ , the composition

$$ko \longrightarrow H \xrightarrow{Sq^r} \Sigma^2 H$$

is trivial, hence it induces the trivial map on  $H_*(ko)$ . Using the Cartan formula for  $Sq_*^2$ , for any element  $w \in H_*(ko)$  we obtain

$$(8.3) \quad \psi_*(w\Sigma^{-1}\zeta_1^2) = w, \quad \psi_*(w\Sigma^{-1}\zeta_2) = w\zeta_1 = w\xi_1, \quad \psi_*(w\Sigma^{-1}\zeta_1^2\zeta_2) = w(\zeta_2 + \zeta_1^3) = w\xi_2.$$

In particular it follows that  $\psi_*: H_*(ko\langle 1 \rangle) \longrightarrow H_{*-1}(H)$  is a monomorphism. We also note that the factorisation of  $\eta: \Sigma ko \longrightarrow ko$  through a  $ko$ -module map  $\tilde{\eta}: \Sigma ko \longrightarrow ko\langle 1 \rangle$  induces

$$\tilde{\eta}_*: H_*(\Sigma ko) \longrightarrow H_*(ko\langle 1 \rangle); \quad \tilde{\eta}_*(w) = w\Sigma^{-1}\zeta_1^2.$$

**Proposition 8.3.** *For any choice of generators  $z_n$  in (8.1), the TAQ-Hurewicz homomorphism  $\theta: \pi_*(MO) \longrightarrow H_*(ko\langle 1 \rangle)$  satisfies*

$$\theta(z_n) = \begin{cases} 0 & \text{if } n \neq 2^s, \\ \Sigma^{-1}\zeta_2 & \text{if } n = 2, \\ \Sigma^{-1}\zeta_1^2\zeta_2 & \text{if } n = 4, \\ \Sigma^{-1}\zeta_1^2\zeta_s & \text{if } n = 2^s \text{ with } s \geq 3. \end{cases}$$

Hence  $MO$  is not a minimal atomic 2-complete commutative  $S$ -algebra.

*Proof.* Choose polynomial generators  $a_n \in H_n(MO)$  so that when  $n + 1$  is not a power of 2,

$$\underline{h}(z_n) = a_n.$$

Note that Kochman's results in [15] give the action of the Dyer-Lashof operations on  $H_*(BO)$  and the Dyer-Lashof indecomposables are spanned by the polynomial generators  $a_{2^s}$  for  $s \geq 0$ . Thus we should only expect  $\theta(z_n)$  to be non-zero when  $n = 2^s$  for some  $s \geq 0$ .

The calculation of  $\psi_* \circ \theta$  require similar methods to those used for  $MU$  in [4, section 3]. The crucial point is the determination of the homomorphism

$$H_*(\mathbb{RP}^\infty) \longrightarrow H_*(BO) \longrightarrow H_*(ko\langle 1 \rangle)$$

induced by the natural inclusion  $\mathbb{RP}^\infty \longrightarrow BO$  and the evaluation

$$\Sigma^\infty BO = \Sigma^\infty \Omega^\infty \Sigma^\infty ko \longrightarrow ko.$$

Composing with  $\psi$  and applying homology we obtain

$$H_*(\mathbb{RP}^\infty) \longrightarrow H_*(ko\langle 1 \rangle) \xrightarrow{\psi_*} H_*(\Sigma H) = \mathcal{A}(2)_{*-1},$$

where  $\psi_*$  is monic. As  $H^1(\mathbb{RP}^\infty) = \mathbb{F}_2$ , our composition is the standard one which maps the generator  $\gamma_n \in H_n(\mathbb{RP}^\infty)$  according to the rule

$$\gamma_n \mapsto \begin{cases} \xi_s & \text{if } n = 2^s - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using (8.3) we see that  $\theta'(a_{2^s})$  has the form claimed.

The statement about  $MO$  not being minimal atomic follows from Thom's result since for each  $s \geq 1$ , there is a homotopy element whose Hurewicz image is  $a_{2^s}$  modulo decomposables in the algebra  $H_*(MO)$ .  $\square$

We end with an argument that is essentially a reformulation of that used in [14, proposition 2.11], some generalisations of which were given in Helen Gilmour's thesis [13]. However here we use computations in  $\text{TAQ}$  rather than directly using Dyer-Lashof operations; the precise connections only become clear in the setting of the Miller-type spectral sequence which we will consider in a planned part II of this work. We will require the following result which appeared in the unpublished preprint of Kriz [16], unpublished work of Basterra and Mandell, and Lazarev [17].

**Theorem 8.4.** *There is an isomorphism*

$$\text{TAQ}^*(H\mathbb{F}_2, S; H\mathbb{F}_2) \cong \mathbb{F}_2\{\Sigma SQ^I : I = (i_1, \dots, i_t) \text{ admissible, } i_t \geq 4\}.$$

Here the symbols  $SQ^I$  behave like the analogous symbols  $Sq^I$  in the Steenrod algebra  $\mathcal{A}(2)^*$ . However, the right hand side should not be regarded as a module over the Steenrod algebra  $\mathcal{A}(2)^*$ , and this is merely an isomorphism of vector spaces. Note the suspension  $\Sigma(-)$  which indicates a degree shift of 1. There is a duality between  $\text{TAQ}^*(H\mathbb{F}_2, S; H\mathbb{F}_2)$  and  $\text{TAQ}_*(H\mathbb{F}_2, S; H\mathbb{F}_2)$ , which says that

$$\text{TAQ}^n(H\mathbb{F}_2, S; H\mathbb{F}_2) \cong \text{Hom}_{\mathbb{F}_2}(\text{TAQ}_n(H\mathbb{F}_2, S; H\mathbb{F}_2), \mathbb{F}_2).$$

In particular we see that

$$\dim_{\mathbb{F}_2} \text{TAQ}_6(H\mathbb{F}_2, S; H\mathbb{F}_2) = \dim_{\mathbb{F}_2} \text{TAQ}^6(H\mathbb{F}_2, S; H\mathbb{F}_2) = 1,$$

where  $\mathrm{TAQ}^6(H\mathbb{F}_2, S; H\mathbb{F}_2)$  is spanned by  $\Sigma SQ^5$ .

**Proposition 8.5.** *There is no morphism of commutative  $S$ -algebras  $H\mathbb{F}_2 \rightarrow MO$ .*

*Proof.* Once again we set  $H = H\mathbb{F}_2$ .

By (8.2),

$$\mathrm{TAQ}_*(MO, S; H) = H_*(ko)\{\Sigma^{-1}\zeta_1^2, \Sigma^{-1}\zeta_2, \Sigma^{-1}\zeta_1^2\zeta_2\},$$

hence  $\mathrm{TAQ}_5(MO, S; H) = 0$ . If such a morphism  $H\mathbb{F}_2 \rightarrow MO$  exists, then since the composition

$$H \xrightarrow{\quad} MO \xrightarrow{\quad} H$$

$\sim$

is the identity on the bottom cell, it must be a weak equivalence since  $H$  is atomic [5]. Applying  $\mathrm{TAQ}_*(-, S; H)$  we obtain a commutative diagram

$$\mathrm{TAQ}_*(H, S; H) \xrightarrow{\quad} \mathrm{TAQ}_*(MO, S; H) \xrightarrow{\quad} \mathrm{TAQ}_*(H, S; H)$$

$\cong$

where  $\mathrm{TAQ}_6(H, S; H) \neq 0$  and  $\mathrm{TAQ}_6(MO, S; H) = 0$ , giving a contradiction.  $\square$

We will not give the details here, but it seems worth mentioning that the Thom spectrum  $MU/O$  associated to the infinite loop space  $U/O$  which is the fibre in the sequence

$$U/O \rightarrow BO \rightarrow BU,$$

is a core for  $MO$ . It turns out that  $H_*(MU/O; \mathbb{F}_2)$  embeds into  $H_*(MO; \mathbb{F}_2)$  as a polynomial subalgebra on odd degree generators the only Dyer-Lashof indecomposable has degree 1. In fact

$$\Omega_S(MU/O) \cong MU/O \wedge \Sigma ko$$

and so

$$\mathrm{TAQ}_*(MU/O, S; H\mathbb{F}_2) = H_*(\Sigma ko; \mathbb{F}_2),$$

and the Dyer-Lashof indecomposable generator maps to  $\Sigma 1$  under the  $\mathrm{TAQ}$ -Hurewicz homomorphism.

#### APPENDIX A. A PROOF OF PROPOSITION 1.6 OF [4]

For completeness we outline a proof of [4, proposition 1.6], due to Philipp Reinhard, unfortunately this was only produced after that paper was published. Our proof, along similar lines to that of McCarthy and Minasian in their proof of [22, theorem 6.1]; unfortunately this appears to be incorrect as stated (at one stage they seem to assume that  $M$  is an algebra).

**Proposition A.1.** *Let  $R$  be a commutative  $S$ -algebra and let  $X$  be a cofibrant  $R$ -module. Then there is a weak equivalence of  $\mathbb{P}_R X$ -modules*

$$\Omega_R(\mathbb{P}_R X) \sim \mathbb{P}_R X \wedge_R X.$$

*Proof.* For every  $M \in \mathcal{M}_{\mathbb{P}_R X}$  there is an adjunction

$$\mathcal{C}_R/\mathbb{P}_R X(\mathbb{P}_R X, \mathbb{P}_R X \vee M) \cong \mathcal{M}_R/\mathbb{P}_R X(X, \mathbb{P}_R X \vee M),$$

where  $\mathcal{M}_R/\mathbb{P}_R X$  denotes the category of  $R$ -modules over  $\mathbb{P}_R X$ . Because the forgetful functor  $\mathcal{C}_R/\mathbb{P}_R X \rightarrow \mathcal{M}_R/\mathbb{P}_R X$  respects fibrations and acyclic fibrations, the adjunction passes to homotopy categories, giving

$$\overline{h}\mathcal{C}_R/\mathbb{P}_R X(\mathbb{P}_R X, \mathbb{P}_R X \vee M) \cong \overline{h}\mathcal{M}_R/\mathbb{P}_R X(X, \mathbb{P}_R X \vee M).$$

Now we have

$$\mathcal{M}_R/\mathbb{P}_R X(\mathbb{P}_R X, M) \cong \mathcal{M}_R/X(X, X \vee M)$$

and the adjunction again passes to homotopy categories and gives

$$\overline{h}\mathcal{M}_R/\mathbb{P}_R X(\mathbb{P}_R X, M) \cong \overline{h}\mathcal{M}_R/X(X, X \vee M).$$

Since in the homotopy category  $X \vee M$  is the product of  $X$  and  $M$ , we have

$$\overline{h}\mathcal{M}_R/X(X, X \vee M) \cong \overline{h}\mathcal{M}_R(X, M).$$

By using the free functor from  $R$ -modules to  $\mathbb{P}_R X$ -modules, we obtain

$$\overline{h}\mathcal{M}_R/X(X, X \vee M) \cong \overline{h}\mathcal{M}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R X, M).$$

Thus we have shown that

$$\begin{aligned} \overline{h}\mathcal{M}_{\mathbb{P}_R X}(\Omega_R(\mathbb{P}_R X), M) &\cong \overline{h}\mathcal{C}_R/\mathbb{P}_R X(\mathbb{P}_R X, \mathbb{P}_R X \vee M) \\ &\cong \overline{h}\mathcal{M}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R X, M). \end{aligned}$$

Using Yoneda's lemma, we obtain the desired equivalence

$$\Omega_R(\mathbb{P}_R X) \sim \mathbb{P}_R X \wedge_R X. \quad \square$$

We also give a useful result on the adjunction for a commutative  $R$ -algebra.

Let  $A$  be a commutative  $R$ -algebra and let  $\tilde{\mu}: \mathbb{P}_R A \rightarrow A$  be the extension of the multiplication. We have

$$\Omega_R(\mathbb{P}_R A) \cong \mathbb{P}_R A,$$

and also the  $A$ -module  $\Omega_R(A)$  becomes a  $\mathbb{P}_R A$ -module via pullback along  $\tilde{\mu}$ . Writing  $\delta$  (without decorations) for universal derivations, we obtain a commutative diagram in  $\overline{h}\mathcal{M}_R$  (with the pentagon commuting in  $\overline{h}\mathcal{M}_{\mathbb{P}_R A}$ ).

$$(A.1) \quad \begin{array}{ccccc} A & \longrightarrow & \mathbb{P}_R A & \xrightarrow{\delta} & \mathbb{P}_R A \wedge_R A \\ & \searrow & \downarrow \tilde{\mu} & & \downarrow \omega(\tilde{\mu}) \\ & & A & \xrightarrow{\delta} & \Omega_R(A) \end{array} \quad \begin{array}{c} \nearrow \tilde{\mu} \wedge \delta \\ \nwarrow \text{mult} \end{array} \quad \begin{array}{c} \\ \\ A \wedge_R \Omega_R(A) \end{array}$$

Here  $\omega(\tilde{\mu})$  denotes the induced ‘derivative’ morphism  $\Omega_R(\mathbb{P}_R A) \rightarrow \Omega_R(A)$ .

**Lemma A.2.** *Suppose that  $M$  is an  $A$ -module and therefore an  $\mathbb{P}_R A$ -module. Then the induced morphism on  $\mathrm{TAQ}_*(-)$ ,  $\tilde{\mu}_*$ , is given by the following commutative diagram.*

$$\begin{array}{ccc}
\mathrm{TAQ}_*(\mathbb{P}_R A, R; M) & \xrightarrow{\tilde{\mu}_*} & \mathrm{TAQ}_*(A, R; M) \\
\parallel & & \parallel \\
\pi_*(M \wedge_R A) & & \pi_*(M \wedge_A \Omega_R(A)) \\
& \searrow (I \wedge \delta_{(A,R)})^* & \nearrow \\
& \pi_*(M \wedge_R \Omega_R(A)) &
\end{array}$$

*Proof.* Apply  $\pi_*(M \wedge_R -)$  to (A.1). □

## APPENDIX B. SOME FORMULAE

We begin by recalling formula due to Kochman [15]. Actually his results are for the infinite loop spaces such as  $BU$ , but the Thom isomorphism commutes with the Dyer-Lashof operations so we will interpret them in the homology of the Thom spectrum  $MU$  with its  $E_\infty$  structure inherited from that of  $BU$ .

Let  $p$  be a prime. We will write  $H_*(-) = H_*(-; \mathbb{F}_p)$ . Let  $b_r \in H_{2r}(MU)$  be the generator obtained as the image of  $\beta_{r+1} \in H_{2r+2}(MU(1)) \cong H_{2r+2}(\mathbb{CP}^\infty)$  under the homomorphism induced by the canonical map  $MU(1) \rightarrow \Sigma^2 MU$  as in [1]. We will use the notation  $x \approx y$  as shorthand for  $x \equiv y \pmod{\text{decomposables}}$ . We also interchangeably use the notations

$$(a, b) = (b, a) = \binom{a+b}{a} = \binom{a+b}{b}$$

for binomial coefficients, where this is taken to be zero if  $a < 0$  or  $b < 0$ . We will use the well-known congruence

$$(B.1) \quad \binom{n_0 + n_1 p + \cdots + n_k p^k}{m_0 + m_1 p + \cdots + m_k p^k} \equiv \binom{n_0}{m_0} \binom{n_0}{m_0} \cdots \binom{n_k}{m_k} \pmod{p}$$

when  $0 \leq m_i, n_i \leq p-1$ .

**Theorem B.1.** *In  $H_*(MU)$  we have*

- if  $p$  is odd,

$$Q^r b_n \approx (-1)^{r+n+1} (n, r-n-1) b_{n+r(p-1)},$$

- if  $p = 2$ ,

$$Q^{2r} b_n \approx (n, r-n-1) b_{n+r}.$$

Note that in the  $p = 2$  case there are analogous results for  $H_*(MO; \mathbb{F}_2)$ .

The Dyer-Lashof operations annihilate 1 and the Cartan formula implies that they act on the indecomposable quotient. In [15, theorem 10], Kochman determined the indecomposable generators which are not in the image of any Dyer-Lashof operations of positive degree. We set

$$Q_{\mathrm{DL}} H_*(MU) = QH_*(MU) / \{Q^s x : s \geq 1, x \in QH_*(MU)\}.$$

**Theorem B.2.** *The indecomposables  $Q_{\mathrm{DL}} H_*(MU)$  have the following elements as a basis:*

- if  $p$  is odd,  $b_{np^t}$  where  $p \nmid n$  and  $n = (\sum_{i=0}^k s_i p^i)(p-1) + r$  with  $r = 1, 2, \dots, p-1$  and if  $\sum_{i=0}^k s_i p^i \neq 0$ ,  $0 \leq s_i \leq (p-1)$  and  $1 \leq s_k \leq s_{k-1} \leq \cdots \leq s_0 \leq r$ ,
- if  $p = 2$ ,  $b_{2^t}$  where  $t \geq 0$ .

The Dyer-Lashof indecomposability of the stated generators can be deduced from our results on the TAQ-Hurewicz homomorphism. As an exercise in computing with binomial coefficients modulo a prime, we have

**Proposition B.3.** *Suppose that  $p$  is a prime and  $n$  has  $p$ -adic expansion*

$$n = n_s p^s + \cdots n_{s+t} p^{s+t}$$

where  $n_s \neq 0 \neq n_{s+t}$  and  $t > 0$ .

If  $p$  is odd then

$$Q^{n-n_s p^s} b_{n_s p^s} \approx \pm \binom{n-n_s p^s-1}{n_s p^s} b_n \not\approx 0.$$

If  $p = 2$  then

$$Q^{2n-2^{s+1}} b_{2^s} \approx \binom{n-2^s-1}{2^s} b_n \not\approx 0.$$

*Proof.* In each case working modulo  $p$  we have

$$\begin{aligned} \binom{n-n_s p^s-1}{n_s p^s} &\equiv \binom{(n-n_s p^s-p^k) + (p^k-1)}{n_s p^s} \\ &\equiv \binom{n-n_s p^s-p^k}{0} \binom{p^k-1}{n_s p^s} \\ &\not\equiv 0, \end{aligned}$$

where  $p^k$  is the highest power of  $p$  dividing  $(n-n_s p^s)$ , and we use the fact that

$$p^k - 1 = (p-1)p^{k-1} + \cdots + (p-1)p^s + \cdots (p-1)p + (p-1)$$

with  $n_s \leq p-1$ . □

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